Optimality-based Bound Contraction with Multiparametric Disaggregation for the Global Optimization of Mixed-Integer Bilinear Problems

Pedro M. Castro[,a] and Ignacio E. Grossmann[b]*

[a] Laboratório Nacional de Energia e Geologia, 1649-038 Lisboa, Portugal

[b] Department of Chemical Engineering, Carnegie Mellon University, Pittsburgh, PA 15213-3890, USA

Abstract

We address nonconvex mixed-integer bilinear problems where the main challenge is the computation of a tight upper bound for the objective function to be maximized. This can be obtained by using the recently developed concept of multiparametric disaggregation following the solution of a mixed-integer linear relaxation of the bilinear problem. Besides showing that it can provide better bounds than a commercial global optimization solver within a given computational time, we propose to also take advantage of the relaxed formulation for contracting the variables domain and further reduce the optimality gap. Through the solution of a real-life case study from a hydroelectric power system, we show that this can be an efficient approach depending on the problem size. The relaxed formulation from multiparametric formulation is provided for a generic numeric representation system featuring a base between 2 (binary) and 10 (decimal).

---

[*] Corresponding author. Tel.: +351-210924643. E-mail: pedro.castro@lneg.pt.

1. Introduction

The problem considered in this paper can be classified as a nonconvex, mixed-integer bilinear program with the following general form:

$$\max z = f_0(x)$$

subject to

$$
\begin{aligned}
&f_q(x) \leq 0 \quad \forall q \in Q\backslash\{0\} \\
&f_q(x) = \sum_{(i,j)\in BL_q} a_{ijq}x_i x_j + h_q(x) \quad \forall q \in Q \\
&x^L \leq x \leq x^U \\
&x \in \mathbb{R}^n, x_m \in \mathbb{Z}, m \in \{1,\dots,r\}
\end{aligned}
\qquad \textbf{(P)}
$$

where x is an n-dimensional vector of non-negative variables that must lie between given lower x^L and upper x^U bounds. The total number of variables is given by n, being $r \leq n$ of the integer type. Set Q includes all functions f_q, including the objective function f_0 and all the constraints. The function $h_q(x)$ is linear in x, BL_q is an (i,j)-index set that defines the bilinear terms $x_i x_j$ present in the problem and a_{ijq} is a scalar. Note that $i \neq j$ for strictly bilinear problems, while $i = j$ can be allowed to accommodate quadratic problems.

The global optimization of mixed-integer nonlinear problem (**P**) is important in areas such as power systems, petroleum blending operations, process networks and production planning. Examples of bilinear functions involving continuous variables are: (i) the production cost term in the objective function of the thermal unit commitment problem [19], which is a quadratic function of power; (ii) the power output in hydro energy systems, related to water discharge and water storage [1,16]; (iii) the properties of a product resulting from a mix of materials, which can be estimated as weighted sums by concentration of the properties of the materials [20]. The binary variables often appear linearly in the formulation and may have different origins: (a) accounting for system operation in multiple consecutive time periods, as in the power systems scheduling problems for the day-ahead electricity markets or in multiperiod blending operations [15]; (b) allowing for connections between units only if the flowrate exceeds a certain minimum value, as in generalized pooling [21-22] or water network design problems [23-24]; (c) choosing between alternative

technologies for treatment units [8,25]. In the trim loss problem [36-37], the bilinear terms involve integer variables related to the number of times a certain paper roll (product) is produced by each cutting pattern and the number of times that each cutting pattern is repeated during the process, appearing in the product demand satisfaction constraint.

Most global optimization approaches for solving bilinear programs rely on the convex McCormick [5] envelopes, which provide a relaxation of the original problem. The quality of the relaxation is highly dependent on the lower and upper bounds of the variables involved in the bilinear terms, improving as their domain is partitioned. This can be done iteratively, as in spatial branch and bound frameworks [18, 26, 31] or simultaneously, using piecewise McCormick envelopes [7-8, 22, 27] or univariate parameterization techniques [10, 28]. An important property of the piecewise McCormick approach of Misener et al. [22] (building on the work of Vielma and Nemhauser [29-30]) and the univariate parameterization techniques, is that the number of binary variables in the mixed-integer linear relaxation grows logarithmically with the number of partitions, leading to an improved computational performance. Changing the type and number of variables involved in the bilinear terms may also improve the quality of the relaxation, in what is known as Reformulation Linearization Technique (RLT) [32]. While Liberti and Pantelides [33] have developed an algorithm that can provide useful insights on how best to formulate a bilinear program, no theoretical or systematic framework exists for deriving RLT formulations with predictably efficient performance.

Multiparametric disaggregation [9] is a univariate parameterization technique that works by discretizing one of the variables of the bilinear term to a specified accuracy level. While initially applied to continuous polynomial problems, it was recently shown that bilinear terms with integer variables are actually a special case of those with real variables [28]. Closely related approaches dealing with bilinear terms featuring binary variables and nonlinear integer problems with signomial terms can be found in [34-35].

The quality of the relaxation from multiparametric disaggregation is dependent on the chosen discretization level but so is problem size. A single increase in the level from p to p-1, measured in terms of powers of ten (10^p) in the decimal numeric representation system, may increase the

complexity so that no reduction in the optimality gap is observed within a reasonable computational time (when compared to p). Provided that modest computational time is required to solve the problem for level p, one can still solve multiple instances of the problem for that same level in a reasonable time.

Regardless of the relaxation approach, the tighter the lower x^L and upper x^U bounds of the variables appearing in the bilinear terms, the higher the quality of the relaxation. Reducing the search space through variable bounding is a feature of most global optimization solvers. GloMIQO [26] for instance, features interval arithmetic, reduced cost and optimality-based bound contraction. The latter involves solving minimization and maximization problems for each variable in the bilinear terms, using the McCormick envelopes [5] for the relaxation.

The main novelty of this paper is to use the relaxation from multiparametric disaggregation [10] to perform optimality-based bound contraction, instead of the standard approach using fixed bounds for the McCormick envelopes [5]. This will enable a further reduction in the domain of the bilinear variables, leading to smaller optimality gaps. The other major difference compared to [10], is that the multiparametric disaggregation formulation is no longer limited to the decimal numeric system for the representation of the discretized variables, allowing the use of a base between 2 and 10. The short-term scheduling problem of hydro power systems will be used as case study.

2.   Upper bounding formulations

In this section, we discuss alternative upper bounding formulations (**PR**) that are a relaxation of (**P**).

2.1.   Using McCormick envelopes

A mixed-integer linear programming (MILP) relaxation can be derived using the McCormick envelopes [5]. It suffices to replace each bilinear term involving variables x_i and x_j, with a new variable $w_{ij} = x_i x_j$, coupled with four sets of constraints. It is easy to check that the bilinear term is feasible in (**PR-MC**) but so are values of w_{ij}, x_i and x_j that do not exactly satisfy $w_{ij} = x_i x_j$. Thus,

(**PR-MC**) is a relaxation of (**P**), the tightest continuous relaxation, for the given lower and upper bounds on variables x_i and x_j, yielding an upper bound for (**P**), i.e. $z^R \geq z$.

$$\max z^R = f_0 = \sum_{(i,j) \in BL_0} a_{ij0} w_{ij} + h_0(x)$$

subject to

$$f_q(x) = \sum_{(i,j) \in BL_q} a_{ijq} w_{ij} + h_q(x) \leq 0 \quad \forall q \in Q \backslash \{0\}$$

$$\left. \begin{array}{l} w_{ij} \geq x_i \cdot x_j^L + x_i^L \cdot x_j - x_i^L \cdot x_j^L \\ w_{ij} \geq x_i \cdot x_j^U + x_i^U \cdot x_j - x_i^U \cdot x_j^U \\ w_{ij} \leq x_i \cdot x_j^L + x_i^U \cdot x_j - x_i^U \cdot x_j^L \\ w_{ij} \leq x_i \cdot x_j^U + x_i^L \cdot x_j - x_i^L \cdot x_j^U \end{array} \right\} \forall\, (i,j) \in BL_q, q \in Q$$

(**PR-MC**)

$$x^L \leq x \leq x^U$$
$$x \in \mathbb{R}^n, x_m \in \mathbb{Z}, m \in \{1, \dots, r\}$$

Remark 1: A tighter MILP relaxation can be obtained through the use of piecewise McCormick envelopes [7-8]. The domain of one of the variables of the bilinear term is partitioned into n disjoint regions, with new binary variables being added to the formulation to select the optimal one. The formulation is capable of generating the global optimal solution to (**P**) for an infinite number of partitions and exhibits a linear increase in problem size with the number of partitions. A somewhat related concept is multiparametric disaggregation [9], which for strictly bilinear terms achieves the same relaxation with a much more favorable logarithmic increase in problem size [10, 12], leading to a considerably better computational performance.

2.2. Using multiparametric disaggregation

Multiparametric disaggregation acts by discretizing x_j over a set of powers $l \in \{p, \dots, P\}$, where $P = \lfloor \log_{10} x_j^U \rfloor$ and p is chosen by the user so as to reach a certain accuracy level. Binary variables $z_{jk\ell}$ identify the digit $k \in \{0, \dots, 9\}$ to select for position l of the decimal numeric representation system (base-10). Since there always exists a gap between discretization points for a finite p, slack variable Δx_j ($0 \leq \Delta x_j \leq 10^p$) is added to obtain a continuous domain.

$$x_j = \sum_{\ell=p}^{P} \sum_{k=0}^{9} 10^\ell \cdot k \cdot z_{jk\ell} + \Delta x_j \quad \forall j \in \{j | (i,j) \in BL_q, q \in Q\}$$

5

The bilinear variable w_{ij} is also written as a sum of approximation terms and a slack variable Δw_{ij}, where $\hat{x}_{ijk\ell}$ is the disaggregated variable linked to x_i and the discrete value k of x_j associated to power l.

$$w_{ij} = \sum_{\ell=p}^{P} \sum_{k=0}^{9} 10^\ell \cdot k \cdot \hat{x}_{ijk\ell} + \Delta w_{ij} \quad \forall\, (i,j) \in BL_q, q \in Q$$

Variables $\hat{x}_{ijk\ell}$ appear naturally from the convex hull reformulation [14] of the disjunctive programming model [13] containing the key constraints of multiparametric disaggregation, as described in [10]. The sum of variables $\hat{x}_{ijk\ell}$ over all digits k must be equal to the original variable x_i for every power l. In case digit k of variable x_j is selected in decimal position l, meaning $z_{jk\ell} = 1$, variable $\hat{x}_{ijk\ell}$ must lie between the upper and lower bounds of variable x_i. Otherwise, $\hat{x}_{ijk\ell} = 0$. Clearly, only one digit k can be associated to power l, thus $\sum_{k=0}^{9} z_{jk\ell} = 1$.

Slack variables Δw_{ij} replace the bilinear terms $x_i \cdot \Delta x_j$ and can be relaxed using McCormick envelopes as described in section 2.1. Although not strictly necessary, we also use the McCormick envelopes on variables w_{ij} to improve the quality of the linear relaxation. The complete MILP formulation (**PR-MDT**) is:

$$\max z^R = f_0 = \sum_{(i,j) \in BL_0} a_{ij0} w_{ij} + h_0(x)$$

subject to

$$f_q(x) = \sum_{(i,j) \in BL_q} a_{ijq} w_{ij} + h_q(x) \leq 0 \quad \forall q \in Q \backslash \{0\}$$

$$w_{ij} = \sum_{\ell=p}^{P} \sum_{k=0}^{9} 10^\ell \cdot k \cdot \hat{x}_{ijk\ell} + \Delta w_{ij} \quad \forall\, (i,j) \in BL_q, q \in Q$$

$$x_j = \sum_{\ell=p}^{P} \sum_{k=0}^{9} 10^\ell \cdot k \cdot z_{jk\ell} + \Delta x_j \quad \forall j \in \{j | (i,j) \in BL_q, q \in Q\} \qquad \textbf{(PR-MDT)}$$

$$x_i = \sum_{k=0}^{9} \hat{x}_{ijk\ell} \quad \forall\, (i,j) \in BL_q, q \in Q, \ell \in L$$

$$x_i^L \cdot z_{jk\ell} \leq \hat{x}_{ijk\ell} \leq x_i^U \cdot z_{jk\ell} \quad \forall\, (i,j) \in BL_q, q \in Q, \ell \in L, k \in K$$

$$\sum_{k=0}^{9} z_{jk\ell} = 1 \quad \forall\, j \in \{j|(i,j) \in BL_q, q \in Q\}, \ell \in L$$

$$\left.\begin{array}{l} x_i^L \cdot \Delta x_j \leq \Delta w_{ij} \leq x_i^U \cdot \Delta x_j \\ \Delta w_{ij} \leq (x_i - x_i^L) \cdot 10^p + x_i^L \cdot \Delta x_j \\ \Delta w_{ij} \geq (x_i - x_i^U) \cdot 10^p + x_i^U \cdot \Delta x_j \end{array}\right\} \forall\, (i,j) \in BL_q, q \in Q$$

$$
\left.\begin{array}{l}
w_{ij} \geq x_i \cdot x_j^L + x_i^L \cdot x_j - x_i^L \cdot x_j^L \\
w_{ij} \geq x_i \cdot x_j^U + x_i^U \cdot x_j - x_i^U \cdot x_j^U \\
w_{ij} \leq x_i \cdot x_j^L + x_i^U \cdot x_j - x_i^U \cdot x_j^L \\
w_{ij} \leq x_i \cdot x_j^U + x_i^L \cdot x_j - x_i^L \cdot x_j^U
\end{array}\right\} \forall\, (i,j) \in BL_q, q \in Q
$$

$$
0 \leq \Delta x_j \leq 10^p \quad \forall\, j \in \{j \mid (i,j) \in BL_q, q \in Q\}
$$

$$
z_{jk\ell} \in \{0,1\} \quad \forall\, j \in \{j \mid (i,j) \in BL_q, q \in Q\}, k \in K, \ell \in L
$$

$$
x^L \leq x \leq x^U
$$
$$
x \in \mathbb{R}^n, x_m \in \mathbb{Z}, m \in \{1, \ldots, r\}
$$

Problem (**PR-MDT**) yields an upper bound z^R for (**P**) that is of a better quality than the one from (**PR-MC**). Furthermore, the value of z^R decreases as p decreases.

Remark 2: Whereas continuous variables being discretized may only assume exact values in the limit $p = -\infty$, for integer variables this will occur for $p = 0$. Thus, if all discretized variables x_j are integer, solving (**PR-MDT**) with $p = 0$ will lead to the global optimal solution of (**P**).

Remark 3: The discretization of variables x_j was done with respect to the decimal numerical representation system but can be extended to a generic b-base coding, $b \in \{2, \ldots, 10\}$. Assuming that the location p of the last significant digit (with respect to the decimal system) is the same for all variables, the number of powers to consider for variable x_j using base b is

$$
L_j^b = \left\lceil \log_b\left(\lfloor x_j^U \cdot 10^{-p}\rfloor + 1\right)\right\rceil
$$

leading to the following definition of x_j:

$$
x_j = 10^p \cdot \sum_{\ell=0}^{L_j^b-1} \sum_{k=1}^{b-1} b^\ell \cdot k \cdot z_{jk\ell} + \Delta x_j \quad \forall j \in \{j \mid (i,j) \in BL_q, q \in Q\}
$$

As an example, for $x_j^U = 0.7374$ and an accuracy of $p = -2$, the number of powers to consider for the binary system is $L_j^2 = 7$. $x_j = 0.7374$ can then be obtained by $10^{-2} \cdot (2^0 + 2^3 + 2^6) + 0.0074$.

Notice that $x_j^U < 10^p \Rightarrow L_{i,t}^b = 0$, meaning $x_j = \Delta x_j$. If this is true for all variables x_j, then the value of z^R from (**PR-MDT**^{b}) will be equal to that from (**PR-MC**).

The MILP formulation (**PR-MDT**^{b}) that provides a relaxation to MINLP problem (**P**) by discretizing the variables using a b-base numeric representation system is then the following:

$$\max z^R = f_0 = \sum_{(i,j) \in BL_0} a_{ij0} w_{ij} + h_0(x)$$

subject to

$$f_q(x) = \sum_{(i,j) \in BL_q} a_{ijq} w_{ij} + h_q(x) \le 0 \quad \forall q \in Q \backslash \{0\}$$

$$w_{ij} = 10^p \cdot \sum_{\ell=0}^{L_j^b-1} \sum_{k=1}^{b-1} b^\ell \cdot k \cdot \hat{x}_{ijk\ell} + \Delta w_{ij} \quad \forall (i,j) \in BL_q, q \in Q$$

$$x_j = 10^p \cdot \sum_{\ell=0}^{L_j^b-1} \sum_{k=1}^{b-1} b^\ell \cdot k \cdot z_{jk\ell} + \Delta x_j \quad \forall j \in \{j|(i,j) \in BL_q, q \in Q\}$$

$$x_i = \sum_{k=0}^{b-1} \hat{x}_{ijk\ell} \quad \forall (i,j) \in BL_q, q \in Q, \ell \in \{0, ..., L_j^b - 1\}$$

$$x_i^L \cdot z_{jk\ell} \le \hat{x}_{ijk\ell} \le x_i^U \cdot z_{jk\ell} \quad \forall (i,j) \in BL_q, q \in Q, k \in \{0, ..., b-1\}, \ell$$
$$\in \{0, ..., L_j^b - 1\}$$

$$\sum_{k=0}^{b-1} z_{jk\ell} = 1 \quad \forall j \in \{j|(i,j) \in BL_q, q \in Q\}, \ell \in \{0, ..., L_j^b - 1\} \qquad \textbf{(PR-MDT}^b\textbf{)}$$

$$\left.\begin{array}{l} x_i^L \cdot \Delta x_j \le \Delta w_{ij} \le x_i^U \cdot \Delta x_j \\ \Delta w_{ij} \le (x_i - x_i^L) \cdot 10^p + x_i^L \cdot \Delta x_j \\ \Delta w_{ij} \ge (x_i - x_i^U) \cdot 10^p + x_i^U \cdot \Delta x_j \end{array}\right\} \forall (i,j) \in BL_q, q \in Q$$

$$\left.\begin{array}{l} w_{ij} \ge x_i \cdot x_j^L + x_i^L \cdot x_j - x_i^L \cdot x_j^L \\ w_{ij} \ge x_i \cdot x_j^U + x_i^U \cdot x_j - x_i^U \cdot x_j^U \\ w_{ij} \le x_i \cdot x_j^L + x_i^U \cdot x_j - x_i^U \cdot x_j^L \\ w_{ij} \le x_i \cdot x_j^U + x_i^L \cdot x_j - x_i^L \cdot x_j^U \end{array}\right\} \forall (i,j) \in BL_q, q \in Q$$

$$0 \le \Delta x_j \le 10^p \quad \forall j \in \{j|(i,j) \in BL_q, q \in Q\}$$

$$z_{jk\ell} \in \{0,1\} \quad \forall j \in \{j|(i,j) \in BL_q, q \in Q\}, k \in \{0, ..., b-1\}, \ell$$
$$\in \{0, ..., L_j^b - 1\}$$

$$x^L \le x \le x^U$$
$$x \in \mathbb{R}^n, x_m \in \mathbb{Z}, m \in \{1, ..., r\}$$

Remark 4: For a strictly bilinear term $x_i x_j$, MDT discretizes variable x_j and disaggregates variable x_i but provides no indication of which variables should be discretized and which should be disaggregated. This is quite relevant when in the presence of bilinear terms from real-life problems, featuring two sets of variables with different domains. As an example, in the short-term hydro scheduling problem considered in this paper, the bilinear terms involve the product of a discharge flowrate by a reservoir volume, and so one can either discretize the flowrate or the volume variables.

Since previous research dealing with other engineering problems [13,15] has shown major differences in computational performance, it is best to try both approaches.

2.2.1. Lower bounding with multiparametric disaggregation

As discussed in [10], the quickest way to obtain a lower bound for problem (**P**) using multiparametric disaggregation is the following. Solve (**PR-MDT**^{b}) and fix the integer variables x_m in (**P**) to the values found by the solution of (**PR-MDT**^{b}), reducing (**P**) to a linear or nonlinear program (NLP). Solve (**P**) with these fixed binary variables using a LP or local NLP algorithm to obtain some z using the solution to (**PR-MDT**^{b}) as a starting point. Note that fixing the binary variables in (**P**) to the values from (**PR-MDT**^{b}) can render (**P**) infeasible. This is more likely to occur when the bilinear terms feature integer variables (e.g. in test problems nvs23 and nvs24 for $p \in \{1,2\}$, see [28]) than when the integer variables appear linearly. In fact, such behavior was not observed for the hydro scheduling problem considered in this paper. In general, it is less likely to occur for smaller values of p, for which (**PR-MDT**^{b}) becomes a better approximation of (**P**).

The computationally more demanding alternative is to remove slack variables Δx_j and Δw_{ij} from (**PR-MDT**^{b}) and solve the resulting MILP problem, as discussed in [10, 28].

3. Bound contraction

The bounds x^L and x^U have a direct influence on the values of z^R obtained by both (**PR-MC**) and (**PR-MDT**). Better bounds will result in a tighter relaxation, leading to values of z^R that are closer to the global optimal solution and hence to smaller optimality gaps. On the other hand, their computation can be very time consuming.

3.1. Standard approach

The standard approach is to solve multiple instances of a problem that is closely related to (**PR-MC**). More precisely, (**BC-MC**) features a different objective function and an additional constraint that limits the domain of the variable under consideration to regions that can actually improve the quality of the lower bound on z, z'. This can be obtained by solving (**P**) with a local MINLP solver.

$$\min/\max x_s$$

subject to

$$f_0(x) = \sum_{(i,j)\in BL_0} a_{ij0}w_{ij} + h_0(x) \geq z'$$

$$f_q(x) = \sum_{(i,j)\in BL_q} a_{ijq}w_{ij} + h_q(x) \leq 0 \quad \forall q \in Q\backslash\{0\} \qquad \textbf{(BC-MC)}$$

$$\left.\begin{array}{l} w_{ij} \geq x_i \cdot x_j^L + x_i^L \cdot x_j - x_i^L \cdot x_j^L \\ w_{ij} \geq x_i \cdot x_j^U + x_i^U \cdot x_j - x_i^U \cdot x_j^U \\ w_{ij} \leq x_i \cdot x_j^L + x_i^U \cdot x_j - x_i^U \cdot x_j^L \\ w_{ij} \leq x_i \cdot x_j^U + x_i^L \cdot x_j - x_i^L \cdot x_j^U \end{array}\right\} \forall\, (i,j) \in BL_q, q \in Q$$

$$x^L \leq x \leq x^U$$

$$x \in \mathbb{R}^n, x_m \in \mathbb{Z}, m \in \{1,\dots,r\}$$

MILP problem (**BC-MC**) is to be solved for every variable s in the bilinear terms, $s \in \{s | (i,j) \in BL_q \wedge (s = i \vee s = j), q \in Q\}$, first minimizing and then maximizing x_s. Following each minimization problem, update the variable's lower bound by making $x_s^L = x_s$, and after a maximization problem, set $x_s^U = x_s$. Since the new bound makes the McCormick envelopes tighter, the chosen sequence for the elements in S affects the final outcome.

Remark 5: To reduce the computational time, the integrality constraints on variables $x_m \in \mathbb{Z}$ can be relaxed so that linear problems (LP) are solved instead. Problem (**BC-MCR**) is obtained from (**BC-MC**) by replacing $x_m \in \mathbb{Z}, m \in \{1,\dots,r\}$ with $x_m \in \mathbb{R}^r$, generating weaker bounds for the variables.

3.2. Novel approach using multiparametric disaggregation

Bound contraction can also be performed with multiparametric disaggregation. It will in general lead to stronger bounds than (**BC-MC**) at the expense of a considerably larger problem, particularly in terms of binary variables due to the inclusion of variables $z_{jk\ell}$. It can be useful in cases where: (i) the increase in (**PR-MDT**^{b}) complexity from discretization level p to $p-1$ is simply too much to improve the value of z^R in a reasonable computational time; and (ii) each (**BC-MDT**^{b}) subproblem can still be solved rather fast for setting p so that (**PR-MDT**^{b}) can benefit from stronger x^L and x^U bounds by returning a lower z^R value and reducing the optimality gap for p. Naturally, one can also apply different p settings for (**BC-MDT**^{b}) and (**PR-MDT**^{b}).

The complete (**BC-MDT^b**) MILP formulation, going along the lines of (**BC-MC**), is given below.

$\min/\max x_s$

subject to

$$f_0(x) = \sum_{(i,j)\in BL_0} a_{ij0} w_{ij} + h_0(x) \geq z'$$

$$f_q(x) = \sum_{(i,j)\in BL_q} a_{ijq} w_{ij} + h_q(x) \leq 0 \quad \forall q \in Q\backslash\{0\}$$

$$w_{ij} = 10^p \cdot \sum_{\ell=0}^{L_j^b-1} \sum_{k=1}^{b-1} b^\ell \cdot k \cdot \hat{x}_{ijk\ell} + \Delta w_{ij} \quad \forall\,(i,j) \in BL_q, q \in Q$$

$$x_j = 10^p \cdot \sum_{\ell=0}^{L_j^b-1} \sum_{k=1}^{b-1} b^\ell \cdot k \cdot z_{jk\ell} + \Delta x_j \quad \forall j \in \{j | (i,j) \in BL_q, q \in Q\}$$

$$x_i = \sum_{k=0}^{b-1} \hat{x}_{ijk\ell} \quad \forall\,(i,j) \in BL_q, q \in Q, \ell \in \{0,\dots,L_j^b-1\}$$

$$x_i^L \cdot z_{jk\ell} \leq \hat{x}_{ijk\ell} \leq x_i^U \cdot z_{jk\ell} \quad \forall\,(i,j) \in BL_q, q \in Q, k \in \{0,\dots,b-1\}, \ell$$
$$\in \{0,\dots,L_j^b-1\} \qquad\qquad \textbf{(BC-MDT}^b\textbf{)}$$

$$\sum_{k=0}^{b-1} z_{jk\ell} = 1 \quad \forall\, j \in \{j | (i,j) \in BL_q, q \in Q\}, \ell \in \{0,\dots,L_j^b-1\}$$

$$\left.\begin{aligned}
x_i^L \cdot \Delta x_j \leq \Delta w_{ij} \leq x_i^U \cdot \Delta x_j \\
\Delta w_{ij} \leq (x_i - x_i^L) \cdot 10^p + x_i^L \cdot \Delta x_j \\
\Delta w_{ij} \geq (x_i - x_i^U) \cdot 10^p + x_i^U \cdot \Delta x_j
\end{aligned}\right\} \forall\,(i,j) \in BL_q, q \in Q$$

$$\left.\begin{aligned}
w_{ij} \geq x_i \cdot x_j^L + x_i^L \cdot x_j - x_i^L \cdot x_j^L \\
w_{ij} \geq x_i \cdot x_j^U + x_i^U \cdot x_j - x_i^U \cdot x_j^U \\
w_{ij} \leq x_i \cdot x_j^L + x_i^U \cdot x_j - x_i^U \cdot x_j^L \\
w_{ij} \leq x_i \cdot x_j^U + x_i^L \cdot x_j - x_i^L \cdot x_j^U
\end{aligned}\right\} \forall\,(i,j) \in BL_q, q \in Q$$

$$0 \leq \Delta x_j \leq 10^p \quad \forall j \in \{j | (i,j) \in BL_q, q \in Q\}$$

$$z_{jk\ell} \in \{0,1\} \quad \forall\, j \in \{j | (i,j) \in BL_q, q \in Q\}, k \in \{0,\dots,b-1\}, \ell$$
$$\in \{0,\dots,L_j^b-1\}$$

$$x^L \leq x \leq x^U$$
$$x \in \mathbb{R}^n, x_m \in \mathbb{Z}, m \in \{1,\dots,r\}$$

4. Short-term hydro scheduling problem

We consider the short-term scheduling problem of a head-sensitive cascaded hydro energy system for the day-ahead electricity market [1]. Given a set $i \in I$ of reservoirs with initial v_i^0, minimum v_i^{min} and maximum v_i^{max} capacities [10^6 m^3], and inlet flow $q_{i,t}^{in}$ [10^6 m^3/h], the objective is to

maximize the total profit of the system. Revenues result from electricity production over the set $t \in T$ of 1-hour periods at a given price c_t [\$/MWh], while the costs are startup costs su_i [\$] introduced to discourage frequent startups that shorten the lifetime of the units as a result of mechanical stress. Power plants are thus not necessarily operating over the full day.

When active, water discharge is subject to given limits q_i^{min} and q_i^{max} [$10^6 \, m^3$/h], as well as power production, p_i^{min} and p_i^{max} [MW]. In addition, the absolute difference between water discharge in two consecutive periods must not exceed q_i^{ramp}, due to requirements of navigation, environment and recreation. Power generation $P_{i,t}$ [MW] at reservoir i is considered a bilinear function of discharge flowrate $Q_{i,t}$ [$10^6 \, m^3$/h] and amount in storage of both i, $V_{i,t}$ [$10^6 \, m^3$], and its immediate downstream reservoir $i' \in I_i$, see eq (1), where α_i, β_i and χ_i are given parameters [2,3]. More accurate power functions involving forebay and tailrace levels, which in turn are calculated as fourth degree polynomials of water storage and discharge flowrate, can be found in [38].

$$P_{i,t} = \left(\chi_i + \alpha_i \beta_i V_{i,t} + \alpha_i \sum_{i' \in I_i} \beta_{i'} V_{i',t} \right) \cdot Q_{i,t} \quad \forall i \in I, t \in T \tag{1}$$

4.1. MINLP formulation

The short-term scheduling problem can be formulated as a discrete-time mixed-integer nonlinear programming formulation [1], where the only source of nonlinearities is eq. (1), with two non-convex bilinear terms. Positive continuous variables include the water flowrate not used for power production (spillage) $S_{i,t}$ [$10^6 \, m^3$/h] and variables $P_{i,t}$, $Q_{i,t}$ and $V_{i,t}$. The binary variables $U_{i,t}$ indicate if reservoir i is producing energy at time t, while binaries $Y_{i,t}$ identify the startup of the plant.

The multiperiod water balance given by eq (2) assumes, due to the short distance between consecutive reservoirs, that the time required for water to travel between consecutive reservoirs is much less than the 1 h time interval considered.

$$V_{i,t} = v_i^0 \big|_{t=1} + V_{i,t-1} + q_{i,t}^{in} + \sum_{\substack{i' \in I \\ i \in I_{i'}}} (Q_{i',t} + S_{i',t}) - Q_{i,t} + S_{i,t} \quad \forall i \in I, t \in T \tag{2}$$

Active power plants must operate within given bounds concerning water discharge and power production, eqs (3)-(4). Notice that $Q_{i,t} = 0 \Leftrightarrow U_{i,t} = 0 \Leftrightarrow P_{i,t} = 0$. As for the reservoirs capacity constraints, they do not feature binary variables, see eq (5).

12

$$q_i^{min} \cdot U_{i,t} \leq Q_{i,t} \leq q_i^{max} \cdot U_{i,t} \quad \forall i \in I, t \in T \tag{3}$$

$$p_i^{min} \cdot U_{i,t} \leq P_{i,t} \leq p_i^{max} \cdot U_{i,t} \quad \forall i \in I, t \in T \tag{4}$$

$$v_i^{min} \leq V_{i,t} \leq v_i^{max} \quad \forall i \in I, t \in T \tag{5}$$

Discharge ramping constraints are given by eq. (6).

$$Q_{i,t} - q_i^{ramp} \leq Q_{i,t+1} \leq Q_{i,t} + q_i^{ramp} \quad \forall i \in I, t \in T\backslash|T| \tag{6}$$

Every time a plant starts operating, we must trigger the startup variables $Y_{i,t}$, which affect the objective function. The required constraint written in logic proposition is $U_{i,t} \wedge \neg U_{i,t-1} \Rightarrow Y_{i,t}$, which can be easily converted to mathematical programming format [4], see eq (7).

$$Y_{i,t} \geq U_{i,t} - U_{i,t-1} \quad \forall i \in I, t \in T \tag{7}$$

The objective is to maximize the profit over a 24-hour period, given by the difference between electricity sales and the startup cost, eq (8).

$$max \sum_{i \in I} \sum_{t \in T} (c_t P_{i,t} - su_i Y_{i,t}) \tag{8}$$

It can easily be seen that the MINLP problem consisting of (1)-(8) is of type (**P**). Solving it to global optimality can be quite difficult as will be seen later on, but a very good feasible solution can be obtained quite fast with a MINLP solver that relies on convexity assumptions. Such a solution will correspond to a lower bound of the global optimal solution.

4.2. Standard MILP relaxation

As seen in section 2.1, a mixed-integer linear programming relaxation of type (**PR-MC**) can easily be derived after defining two new sets of continuous variables $W_{i,t}^1 = Q_{i,t}V_{i,t}$ and $W_{i',i,t}^2 = Q_{i,t}V_{i',t}$ to replace the bilinear terms, and adding new sets of constraints. Linear eq (9) replaces nonlinear eq (1) and eqs (10)-(17) are the McCormick envelopes [5]. The solution of the MILP problem consisting of eqs (2)-(17) will provide an upper bound on the profit that can be obtained over a 24-hour period. Relaxing the integrality constraints on variables $U_{i,t}$ and $Z_{i,t}$ will lead to a weaker upper bound resulting from the solution of a LP.

$$P_{i,t} = \chi_i + \alpha_i \beta_i W_{i,t}^1 + \alpha_i \sum_{i' \in I_i} \beta_{i'} W_{i',i,t}^2 \quad \forall i \in I, t \in T \tag{9}$$

$$W_{i,t}^1 \geq V_{i,t} q_{i,t}^L + v_{i,t}^L Q_{i,t} - v_{i,t}^L q_{i,t}^L \quad \forall i \in I, t \in T \tag{10}$$

$$W_{i,t}^1 \geq V_{i,t}q_{i,t}^U + v_{i,t}^U Q_{i,t} - v_{i,t}^U q_{i,t}^U \quad \forall i \in I, t \in T \tag{11}$$

$$W_{i,t}^1 \leq V_{i,t}q_{i,t}^L + v_{i,t}^U Q_{i,t} - v_{i,t}^U q_{i,t}^L \quad \forall i \in I, t \in T \tag{12}$$

$$W_{i,t}^1 \leq V_{i,t}q_{i,t}^U + v_{i,t}^L Q_{i,t} - v_{i,t}^L q_{i,t}^U \quad \forall i \in I, t \in T \tag{13}$$

$$W_{i',i,t}^2 \geq V_{i',t}q_{i,t}^L + v_{i',t}^L Q_{i,t} - v_{i',t}^L q_{i,t}^L \quad \forall i \in I, i' \in I_i, t \in T \tag{14}$$

$$W_{i',i,t}^2 \geq V_{i',t}q_{i,t}^U + v_{i',t}^U Q_{i,t} - v_{i',t}^U q_{i,t}^U \quad \forall i \in I, i' \in I_i, t \in T \tag{15}$$

$$W_{i',i,t}^2 \leq V_{i',t}q_{i,t}^L + v_{i',t}^U Q_{i,t} - v_{i',t}^U q_{i,t}^L \quad \forall i \in I, i' \in I_i, t \in T \tag{16}$$

$$W_{i',i,t}^2 \leq V_{i',t}q_{i,t}^U + v_{i',t}^L Q_{i,t} - v_{i',t}^L q_{i,t}^U \quad \forall i \in I, i' \in I_i, t \in T \tag{17}$$

The lower and upper bounds on variables $Q_{i,t}$ and $V_{i,t}$ are given by parameters $q_{i,t}^L$, $q_{i,t}^U$, $v_{i,t}^L$ and $v_{i,t}^U$, respectively. These feature the reservoir as well as the time index to highlight that strong values can be calculated through the bound contraction procedures described in sections 3.1 and 3.2, involving the solution of problems of type (**BC-MC**) or (**BC-MDT^b**). One can alternatively ignore the time index and rely on the given capacity and discharge flow limits, see eqs (18)-(20), which are also involved in the initialization of the bound contraction procedure. Notice in eq. (20) that the lower bound on the discharge variables cannot be made equal to the minimum discharge flow q_i^{min} since such constraint is to be enforced only if reservoir i is producing energy.

$$v_{i,t}^L = v_i^{min} \quad \forall i \in I, t \in T \tag{18}$$

$$v_{i,t}^U = v_i^{max} \quad \forall i \in I, t \in T \tag{19}$$

$$q_{i,t}^L = 0 \quad \forall i \in I, t \in T \tag{20}$$

$$q_{i,t}^U = q_i^{max} \quad \forall i \in I, t \in T \tag{21}$$

4.3. MILP relaxation using multiparametric disaggregation

We now provide the full set of constraints corresponding to (**PR-MDT^b**) for the hydro scheduling problem, together with the basic building block, where the main concept is expressed as a disjunctive program [13]. Of the two sets of variables appearing in bilinear terms, we can either discretize: (i) water discharge flowrate; or (ii) volume in storage. Both alternatives are considered.

4.3.1. Discretizing the discharge flowrate variables $Q_{i,t}$

The multiparametric disaggregation technique builds on the fact that a given variable can be calculated as a sum of approximation $\tilde{Q}_{i,t}$ and residual terms $\Delta Q_{i,t}$, see eqs 24-26.

$$\underbrace{Q_{i,t}}_{l\in\mathbb{N}} = \underbrace{\tilde{Q}_{i,t}}_{l\in\{0,\dots,L_{i,t}^b-1\}} + \underbrace{\Delta Q_{i,t}}_{l\in\{L_{i,t}^b,\dots,+\infty\}} \tag{24}$$

$$L_{i,t}^b = \left\lceil \log_b\left(\lfloor q_{i,t}^U \cdot 10^{-p}\rfloor + 1\right)\right\rceil \ \forall i \in I, t \in T \tag{25}$$

$$0 \leq \Delta Q_{i,t} \leq 10^p \ \forall i \in I, t \in T \tag{26}$$

Similarly, the exact representation of bilinear term variables $W_{i,t}^1$ and $W_{i',i,t}^2$ can be divided into approximate and residual terms, see eqs (27)-(28).

$$W_{i,t}^1 = V_{i,t} \cdot \left(\tilde{Q}_{i,t} + \Delta Q_{i,t}\right) = V_{i,t} \sum_{l=0}^{L_{i,t}^b-1} \tilde{Q}_{i,t,l} + V_{i,t}\Delta Q_{i,t} = \sum_{l=0}^{L_{i,t}^b-1} \widetilde{W}_{i,t,l}^1 + \Delta W_{i,t}^1 = \widetilde{W}_{i,t}^1 +$$

$$\Delta W_{i,t}^1 \ \forall i \in I, t \in T \tag{27}$$

$$W_{i',i,t}^2 = V_{i',t} \cdot \left(\tilde{Q}_{i,t} + \Delta Q_{i,t}\right) = V_{i',t} \sum_{l=0}^{L_{i,t}^b-1} \tilde{Q}_{i,t,l} + V_{i',t}\Delta Q_{i,t} = \sum_{l=0}^{L_{i,t}^b-1} \widetilde{W}_{i',i,t,l}^2 + \Delta W_{i',i,t}^2 = \widetilde{W}_{i',i,t}^2 +$$

$$\Delta W_{i',i,t}^2 \ \forall i \in I, i' \in I_i, t \in T \tag{28}$$

The problem of finding the value of the approximate variables for position l, can actually be modeled as a disjunctive program. Considering all possible parameters $k \in \{0, \dots, b-1\}$ and making sure a single one (exclusive OR) is selected through the use of Boolean variables $Z_{i,t,k,l}$, eq (29) results.

$$\bigvee_{k=0}^{b-1} \begin{bmatrix} Z_{i,t,k,l} \\ \tilde{Q}_{i,t,l} = 10^p \cdot k \cdot b^l \\ \widetilde{W}_{i,t,l}^1 = 10^p \cdot V_{i,t} \cdot k \cdot b^l \\ \widetilde{W}_{i',i,t,l}^2 = 10^p \cdot V_{i',t} \cdot k \cdot b^l \ \forall i' \in I_i \end{bmatrix} \forall i \in I, t \in T, l \in \{0, \dots, L_{i,t}^b - 1\} \tag{29}$$

The convex hull reformulation [14] of eq (29) followed by a few algebraic transformations of constraints to avoid the use of approximation variables $\tilde{Q}_{i,t}, \tilde{Q}_{i,t,l}, \widetilde{W}_{i,t}^1, \widetilde{W}_{i',i,t}^2, \widetilde{W}_{i,t,l}^1$ and $\widetilde{W}_{i',i,t,l}^2$, leads us to the majority of the constraints of the upper bounding formulation, see eqs (30)-(41). Notice that the Boolean variables $Z_{i,t,k,l}$ have been converted to binary variables.

$$W_{i,t}^1 = 10^p \cdot \sum_{l=0}^{L_{i,t}^b-1} \sum_{k=0}^{b-1} \hat{V}_{i,t,k,l}^1 \cdot k \cdot b^l + \Delta W_{i,t}^1 \ \forall i \in I, t \in T \tag{30}$$

$$W_{i',i,t}^2 = 10^p \cdot \sum_{l=0}^{L_{i,t}^b-1} \sum_{k=0}^{b-1} \hat{V}_{i',i,t,k,l}^2 \cdot k \cdot b^l + \Delta W_{i',i,t}^2 \ \forall i \in I, i' \in I_i, t \in T \tag{31}$$

15

$$V_{i,t} = \sum_{k=0}^{b-1} \hat{V}^1_{i,t,k,l} \quad \forall i \in I, t \in T, l \in \{0, \dots, L^b_{i,t} - 1\} \tag{32}$$

$$V_{i',t} = \sum_{k=0}^{b-1} \hat{V}^2_{i',i,t,k,l} \quad \forall i \in I, i' \in I_i, t \in T, l \in \{0, \dots, L^b_{i,t} - 1\} \tag{33}$$

$$v^L_{i,t} \cdot Z_{i,t,k,l} \leq \hat{V}^1_{i,t,k,l} \leq v^U_{i,t} \cdot Z_{i,t,k,l} \quad \forall i \in I, t \in T, k \in \{0, \dots, b-1\}, l \in \{0, \dots, L^b_{i,t} - 1\} \tag{34}$$

$$v^L_{i',t} \cdot Z_{i,t,k,l} \leq \hat{V}^2_{i',i,t,k,l} \leq v^U_{i',t} \cdot Z_{i,t,k,l} \quad \forall i \in I, i' \in I_i, t \in T, k \in \{0, \dots, b-1\}, l \in \{0, \dots, L^b_{i,t} - 1\} \tag{35}$$

$$\sum_{k=0}^{b-1} Z_{i,t,k,l} = 1 \quad \forall i \in I, t \in T, l \in \{0, \dots, L^b_{i,t} - 1\} \tag{36}$$

$$Q_{i,t} = 10^p \cdot \sum_{l=0}^{L^b_{i,t}-1} \sum_{k=0}^{b-1} k \cdot b^l \cdot Z_{i,t,k,l} + \Delta Q_{i,t} \quad \forall i \in I, t \in T \tag{37}$$

$$q^L_{i,t} \leq Q_{i,t} \leq q^U_{i,t} \quad \forall i \in I, t \in T \tag{38}$$

$$\hat{V}^1_{i,t,k,l} \geq 0 \quad \forall i \in I, t \in T, k \in \{0, \dots, b-1\}, l \in \{0, \dots, L^b_{i,t} - 1\} \tag{39}$$

$$\hat{V}^2_{i',i,t,j,k} \geq 0 \quad \forall i \in I, i' \in I_i, t \in T, k \in \{0, \dots, b-1\}, l \in \{0, \dots, L^b_{i,t} - 1\} \tag{40}$$

$$Z_{i,t,k,l} = \{0,1\} \quad \forall i \in I, t \in T, k \in \{0, \dots, b-1\}, l \in \{0, \dots, L^b_{i,t} - 1\} \tag{41}$$

It remains dealing with the residual variables $\Delta W^1_{i,t}$ and $\Delta W^2_{i',i,t}$. They involve bilinear terms themselves that can be relaxed using the standard McCormick envelopes [5] discussed in section 2.1. Using the bounds for $\Delta Q_{i,t}$ given in eq. (26), eqs (42)-(45) can be obtained.

$$v^L_{i,t} \cdot \Delta Q_{i,t} \leq \Delta W^1_{i,t} \leq v^U_{i,t} \cdot \Delta Q_{i,t} \quad \forall i \in I, t \in T \tag{42}$$

$$(V_{i,t} - v^U_{i,t}) \cdot 10^p + v^U_{i,t} \cdot \Delta Q_{i,t} \leq \Delta W^1_{i,t} \leq (V_{i,t} - v^L_{i,t}) \cdot 10^p + v^L_{i,t} \cdot \Delta Q_{i,t} \quad \forall i \in I, t \in T \tag{43}$$

$$v^L_{i',t} \cdot \Delta Q_{i,t} \leq \Delta W^2_{i',i,t} \leq v^U_{i',t} \cdot \Delta Q_{i,t} \quad \forall i \in I, t \in T \tag{44}$$

$$(V_{i',t} - v^U_{i',t}) \cdot 10^p + v^U_{i',t} \cdot \Delta Q_{i,t} \leq \Delta W^2_{i',i,t} \leq (V_{i',t} - v^L_{i',t}) \cdot 10^p + v^L_{i',t} \cdot \Delta Q_{i,t} \quad \forall i \in I, t \in T \tag{45}$$

The complete MILP formulation that provides an upper bound to the original MINLP problem is thus given by eqs (2)-(17), (25)-(26), (30)-(45).

4.3.2. Discretizing the volume in storage variables $V_{i,t}$

The second option is to discretize variables $V_{i,t}$. Keeping the same nomenclature and the bounds on the residual variables, which are only a function of p, the required number of positions for the chosen accuracy is now a function of the volume in storage upper bound, see eqs (46)-(47).

$$0 \leq \Delta V_{i,t} \leq 10^p \quad \forall i \in I, t \in T \tag{46}$$

$$L_{i,t}^b = \left\lceil \log_b(\lfloor v_{i,t}^U \cdot 10^{-p} \rfloor + 1) \right\rceil \ \forall i \in I, t \in T \tag{47}$$

The binary variables $Z_{i,t,k,l}$ now identify the active digit k at position l for the representation (in base b) of the volume of reservoir i at time t. To better understand the disjunctive program given in eq (48), note that variable $V_{i,t}$ appears in the power definition constraint (eq. 1) of reservoir i and its upstream reservoir i' ($i \in I_{i'}$). Hence, the change in the order of indices of variable $\widetilde{W}_{i,i',t,l}^2$ (compare to eq 29).

$$\underset{k=0}{\overset{b-1}{\vee}} \begin{bmatrix} Z_{i,t,k,l} \\ \widetilde{V}_{i,t,l} = 10^p \cdot k \cdot b^l \\ \widetilde{W}_{i,t,l}^1 = 10^p \cdot Q_{i,t} \cdot k \cdot b^l \\ \widetilde{W}_{i,i',t,l}^2 = 10^p \cdot Q_{i',t} \cdot k \cdot b^l \ \forall i' \in I \wedge i \in I_{i'} \end{bmatrix} \forall i \in I, t \in T, l \in \{0, \dots, L_{i,t}^b - 1\} \tag{48}$$

Applying the convex hull reformulation leads to the complete set of MILP constraints given below in eqs (49)-(62). Eqs (36), (41) and (46)-(47), also apply.

$$W_{i,t}^1 = 10^p \cdot \sum_{l=0}^{L_{i,t}^b-1} \sum_{k=0}^{b-1} \hat{Q}_{i,t,k,l}^1 \cdot k \cdot b^l + \Delta W_{i,t}^1 \ \forall i \in I, t \in T \tag{49}$$

$$W_{i,i',t}^2 = 10^p \cdot \sum_{l=0}^{L_{i,t}^b-1} \sum_{k=0}^{b-1} \hat{Q}_{i,i',t,k,l}^2 \cdot k \cdot b^l + \Delta W_{i,i',t}^2 \ \forall i \in I, i' \in I \wedge i \in I_{i'}, t \in T \tag{50}$$

$$Q_{i,t} = \sum_{k=0}^{b-1} \hat{Q}_{i,t,k,l}^1 \ \forall i \in I, t \in T, l \in \{0, \dots, L_{i,t}^b - 1\} \tag{51}$$

$$Q_{i',t} = \sum_{k=0}^{b-1} \hat{Q}_{i,i',t,k,l}^2 \ \forall i \in I, i' \in I \wedge i \in I_{i'}, t \in T, l \in \{0, \dots, L_{i,t}^b - 1\} \tag{52}$$

$$q_{i,t}^L \cdot Z_{i,t,k,l} \leq \hat{Q}_{i,t,k,l}^1 \leq q_{i,t}^U \cdot Z_{i,t,k,l} \ \forall i \in I, t \in T, k \in \{0, \dots, b-1\}, l \in \{0, \dots, L_{i,t}^b - 1\} \tag{53}$$

$$q_{i',t}^L \cdot Z_{i,t,k,l} \leq \hat{Q}_{i,i',t,k,l}^2 \leq q_{i',t}^U \cdot Z_{i,t,k,l} \ \forall i \in I, i' \in I \wedge i \in I_{i'}, t \in T, k \in \{0, \dots, b-1\}, l \in$$
$$\{0, \dots, L_{i,t}^b - 1\} \tag{54}$$

$$V_{i,t} = 10^p \cdot \sum_{l=0}^{L_{i,t}^b-1} \sum_{k=0}^{b-1} k \cdot b^l \cdot Z_{i,t,k,l} + \Delta V_{i,t} \ \forall i \in I, t \in T \tag{55}$$

$$v_{i,t}^L \leq V_{i,t} \leq v_{i,t}^U \ \forall i \in I, t \in T \tag{56}$$

$$q_{i,t}^L \cdot \Delta V_{i,t} \leq \Delta W_{i,t}^1 \leq q_{i,t}^U \cdot \Delta V_{i,t} \ \forall i \in I, t \in T \tag{57}$$

$$(Q_{i,t} - q_{i,t}^U) \cdot 10^p + q_{i,t}^U \cdot \Delta V_{i,t} \leq \Delta W_{i,t}^1 \leq (Q_{i,t} - q_{i,t}^L) \cdot 10^p + q_{i,t}^L \cdot \Delta V_{i,t} \ \forall i \in I, t \in T \tag{58}$$

$$q_{i',t}^L \cdot \Delta V_{i,t} \leq \Delta W_{i,i',t}^2 \leq q_{i',t}^U \cdot \Delta V_{i,t} \ \forall i \in I, i' \in I \wedge i \in I_{i'}, t \in T \tag{59}$$

$$(Q_{i',t} - q_{i',t}^U) \cdot 10^p + q_{i',t}^U \cdot \Delta V_{i,t} \leq \Delta W_{i,i',t}^2 \leq (Q_{i',t} - q_{i',t}^L) \cdot 10^p + q_{i',t}^L \cdot \Delta V_{i,t} \ \forall i \in I, i' \in I \wedge i \in$$
$$I_{i'}, t \in T \tag{60}$$

$$\hat{Q}^1_{i,t,k,l} \geq 0 \;\; \forall i \in I, t \in T, k \in \{0, \dots, b-1\}, l \in \{0, \dots, L^b_{i,t} - 1\} \tag{61}$$

$$\hat{Q}^2_{i,i',t,k,l} \geq 0 \;\; \forall i \in I, i' \in I \wedge i \in I_{i'}, t \in T, k \in \{0, \dots, b-1\}, l \in \{0, \dots, L^b_{i,t} - 1\} \tag{62}$$

4.4. Problem data

The hydro energy system under consideration is located in the Douro River, in Portugal, and consists of seven cascaded reservoirs [1], see Figure 1. Hydro plants $i \in \{1,2,4,5,7\}$ are run-of-the-river, characterized by a small storage capacity and an operating efficiency that is sensitive to the head change effect [2,3]. On the other hand, plants $i \in \{3,6\}$ are storage hydro plants. Inflow $q^{in}_{i,t}$ is considered only on reservoirs 1-6 and the final water storage in the reservoirs is constrained to be equal to the initial water storage: $V_{i,|T|} = v^0_i$. Startup costs are a linear function of maximum power production [16]: $su_i = 2.5 p^{max}_i$ [\$/MW]. Model parameters that can be revealed are given in Table 1, while the electricity price profile is given in Table 2.

Table 1. Hydro system data

i	v^0_i [10^6 m^3]	v^{min}_i [10^6 m^3]	v^{max}_i [10^6 m^3]	p^{min}_i [MW]	p^{max}_i [MW]	q^{min}_i [10^6 m^3/h]	q^{max}_i [10^6 m^3/h]
1	10.35	5.18	12.94	28	188.08	0.60527	4.12020
2	10.64	5.32	13.3	29.99	237.14	0.37692	3.88800
3	78	39	97.5	10.64	60	0.01080	0.05904
4	9.6	4.8	12	24.99	185.99	0.37681	3.24000
5	8.8	4.4	11	29.99	201.02	0.33563	3.17272
6	46.7	36.89	58.38	39.99	134.02	0.34196	1.17482
7	17.2	8.6	21.5	19.99	117.01	0.65819	4.88344

Table 2. Electricity price profile [\$/MWh]

Hour	0	1	2	3	4	5	6	7
c_t	50.400	46.287	44.187	44.787	45.477	47.523	58.359	68.487
Hour	8	9	10	11	12	13	14	15
c_t	87.441	91.395	93.846	94.995	86.187	92.295	93.495	92.259
Hour	16	17	18	19	20	21	22	23
c_t	93.795	103.254	103.359	100.623	95.418	92.136	82.305	68.946
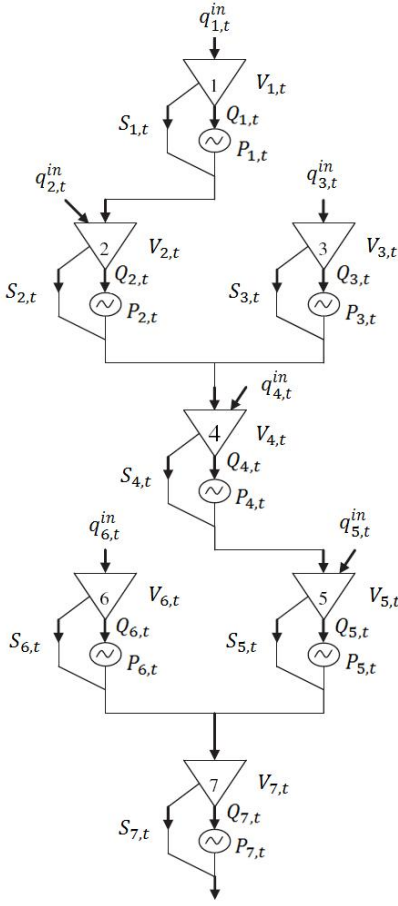
Figure 1: Hydro energy system under consideration.

5. Numerical experiments

The short-term hydro scheduling problem of type (**P**) is tackled with commercial MINLP solvers DICOPT, an outer approximation algorithm [17] guaranteed to converge only under certain convexity assumptions, and BARON [18], a branch-and-bound algorithm enhanced with a variety of constraint propagation and duality techniques that can provide global optimal solutions. MILP formulations resulting from the relaxation of the bilinear terms using McCormick envelopes (**PR-MC**) or multiparametric disaggregation (**PR-MDT^b**) are solved with CPLEX. These are capable of providing upper bounds on (**P**) that can be used to compute an optimality $gap = 1 - z/z^r$. Problems of type (**PR-MDT^b**) are generated by discretizing over variables $Q_{i,t}$ and $V_{i,t}$, for different discretization levels p and basis $b \in \{2,10\}$.

Since the quality of the upper bound is greatly influenced by the quality of the variables bounds, problems (**PR-MC**) and (**PR-MDT^b**) are solved following three different bound contraction strategies: (i) no bound contraction, with bounds on the variables calculated from eqs (18)-(20); (ii)

19

standard bound contraction using McCormick envelopes (**BC-MC**); (iii) new strategy using multiparametric disaggregation (**BC-MDT^b**). We also provide the lower bound resulting from the solution of the NLP with commercial solver CONOPT, after fixing the binary values of (**P**) to the values obtained from multiparametric disaggregation (**PR-MDT^b**), recall section 2.2.1.

In order to evaluate the influence of problem size on performance, smaller problems consisting of the first 2 and 4 reservoirs are considered besides the full, 7-reservoir problem. All mathematical formulations and algorithms were implemented in GAMS 23.8.2 and solved on an Intel i7 950 (3.07 GHz) desktop computer with 8 GB of RAM and running Windows 7. Default options were used for the commercial solvers and the maximum computational time for each solve statement was set to 3600 CPUs unless stated otherwise. MILP problems were tacked using a single thread down to a 10^{-6} relative optimality tolerance.

5.1.   Optimal solution for problem (P)

We start by analyzing the best feasible solution returned by the commercial solvers. As can be seen in Table 3, the optimal solutions (*z*) from DICOPT are better than the ones obtained by BARON in 1-hour of computational time, even after performing bound contraction. It is a clear indication that DICOPT is a better solver for the solution of the hydro scheduling problem for the day-ahead electricity market, requiring just a couple of seconds to provide very good solutions. In fact, we were unable to improve the values listed in the 2nd column using the methods described in this paper even after several computational runs, often terminating with suboptimal solutions.

Table 3. Results for (P) using commercial MINLP solvers

|*I*|	DICOPT		BARON			BARON preceded by (**BC-MCR**)				MINLP problem size		
	z [$]	CPUs	*z* [$]	z^R [$]	*gap*[a]	*z* [$]	z^R [$]	*gap*[a]	CPUs	Binary variables	Total Variables	Equations
2	**209721**	0.59	209712	210663	0.45%	209712	210547	0.39%	3600[b]	96	289	429
4	**371812**	1.06	371737	376560	1.26%	371195	376120	1.15%	3600[b]	192	577	857
7	**744964**	2.13	743432	761999	2.24%	743752	760461	2.04%	3600[b]	336	1009	1499

[a] Optimality gap calculated using *z* from DICOPT. [b] Maximum resource limit.

The focus of the paper is however on determining whether such solutions are indeed globally optimal, which is done by exploring systematic solution strategies that are capable of reducing the optimality gap. A good reference for comparison purposes can be calculated using the upper bound

(z^R) from BARON at the time of termination. Notice that the optimality gap increases 5 times when going from the 2 to the 7-reservoir problem.

5.2. Upper bound from McCormick envelopes

A fast way of determining an upper bound to (**P**) is to relax the bilinear terms using the McCormick envelopes and solve MILP problem (**PR-MC**) or its relaxed LP version. The results in Table 4 show that it is advantageous to consider the MILP since lower gaps are obtained at almost no additional computational cost. These can be further reduced by first tightening the lower and upper bounds of the two sets of model variables forming the bilinear terms. It involves solving $|I| \cdot |T| \cdot 2 \cdot 2 = 96 \cdot |I|$ problems of type (**BC-MC**), or (**BC-MCR**) in case the integrality constraints are dropped, using the optimal solution from DICOPT as z'. Indeed, (**BC-MCR**) is a better option for bound contraction since the results show a minor improvement in terms of optimality gap for (**BC-MC**), for up to 5.8 times increase in computational time, which is already of the order of minutes.

Table 4. Results for (PR-MC) with and without bound contraction

	(PR-MC) LP			(PR-MC)			(BC-MCR) + (PR-MC)			(BC-MC) + (PR-MC)		
$|I|$	z^R [$]	gap^a	CPUs	z^R [$]	gap^a	CPUs	z^R [$]	gap^a	CPUs	z^R [$]	gap^a	CPUs
2	216391	3.08%	0.12	215703	2.77%	0.25	214128	2.06%	22.7	214040	2.02%	54.1
4	383107	2.95%	0.12	382323	2.75%	0.37	379743	2.09%	50.6	379724	2.08%	176
7	770942	3.37%	0.15	770108	3.27%	0.60	764098	2.50%	108	764087	2.50%	627

[a] Optimality gap calculated using z from DICOPT.

5.3. On the choice of discretized variables and discretization level

Multiparametric disaggregation can improve the quality of the upper bound by discretizing the domain of half of the variables of the bilinear terms. Yet, decisions need to be made concerning: (i) which variables to discretize, $Q_{i,t}$ or $V_{i,t}$; (ii) the appropriate discretization level p. Concerning (i) both alternatives are tested, while for (ii) we start with the maximum value still leading to discretization of some variables. According to Table 1, the upper bounds on the discharge (q_i^{max}) and volume (v_i^{max}) variables are at most 4.88 and 97.5 (using the appropriate units) and so as discussed in section 2.2, we start with $p = 0$ for $Q_{i,t}$ and $p = 1$ for $V_{i,t}$. Provided that (**PR-MDT**[10])

for p is solved to optimality before the 1-hour mark, we also solve it for $p = p - 1$. Following the solution of (**PR-MDT^{10}**), binary variables $U_{i,t}$ and $Y_{i,t}$ are fixed to the values obtained so that (**P**) becomes a NLP and a good feasible solution z can be found.

In Table 5, we give the results without bound contraction, where it can be seen that the lowest gaps are equal to 0.21, 0.97 and 1.95% (compare to the 0.45, 1.26 and 2.24% returned by BARON in Table 3), for $|I| = 2,4,7$, respectively. Reducing the domain of the variables through bound contraction (**BC-MCR**), has the double advantage of making the formulation tighter and reducing the number of binary variables of type $Z_{i,t,k,l}$. It takes at most 108 CPUs (recall Table 4) and leads to a significant improvement in the values for $|I| = 4,7$. The gaps now become 0.20, 0.73 and 1.62 % (compare to 0.39, 1.15 and 2.04% by BARON), see Table 6. Furthermore, we could find DICOPT's optimal solution for $|I| = 2$ and all runs for $|I| = 7$ provided better schedules than the best one found by BARON, which solves multiple rather than a single NLP subproblem in the search for the global optimal solution. These results confirm the potential of multiparametric disaggregation to provide tighter bounds and better solutions than those obtained by state-of-the-art solvers in the same computational time.

Another interesting result from Table 5 and Table 6 is that the appropriate setting related to decisions (i) and (ii) varies significantly with the problem size. More specifically, the lowest gaps were obtained for: $V_{i,t}$ and $p = -1$ ($|I| = 2$); $Q_{i,t}$ and $p = 0$ or $V_{i,t}$ and $p = 0$ ($|I| = 4$); $Q_{i,t}$ and $p = 0$ or $V_{i,t}$ and $p = 1$ ($|I| = 7$). For the given q_i^{max} and v_i^{max} values, one can argue that the approximation of the discretized variables becomes more accurate in the sequence: $V_{i,t}, p = 1 \rightarrow Q_{i,t}, p = 0 \rightarrow V_{i,t}, p = 0 \rightarrow Q_{i,t}, p = -1 \rightarrow V_{i,t}, p = -1$ (higher accuracy is somewhat related to more binary variables, see computational statistics in Table 7). This does not necessarily mean that the optimality gap always decreases from left to right since the increase in accuracy also augments problem size, and so a tradeoff is involved. As an example, the results in Table 5 for $|I| = 4$ show a higher gap (1.29%) for $V_{i,t}, p = 0$ than $Q_{i,t}, p = 0$ (0.97%), but as soon as we make the formulations tighter due to bound contraction, the MILP solver becomes more efficient, with the results showing a lower gap at termination for $V_{i,t}, p = 0$ (0.73% vs. 0.80%, see Table 6). The overall

recommendation is thus to test both discretization alternatives to take the most of multiparametric disaggregation in the available computational time.

5.4. Basis for numeric representation

Multiparametric disaggregation can be applied to basis other than 10. In Table 8, we provide the results of (**PR-MDT^2**) after bound contraction, where a similar performance to (**PR-MDT^{10}**) can be observed. Nevertheless, we were now able to find DICOPT's optimal solution for $|I| = 4$ ($V_{i,t}$, $p = 0$), while reducing the gap to 0.69%. The optimality gap for $|I| = 7$ was also slightly reduced to 1.60% ($V_{i,t}$, $p = 1$). Another aspect worth highlighting is that the value of z^R is independent of the chosen basis, provided that the MILP problem is solved to optimality.

Comparing the computational statistics related to problem size (Table 9 with Table 7) it can be observed that the binary base is generating problems with a larger number of variables and constraints, contrary to previous observations [11]. It can be explained by the lower discretization level used in this work and the tight variable domains, since increasing the value of p will result in more binary variables being required by base 10 (increase in k prevails over decrease in l) and more constraints being required by base 2 (check $V_{i,t}$, $p = -1$ columns). Using the 4 reservoir problem as an example, after bound contraction we get $v_{3,1}^l = 77.91$ and $v_{3,1}^U = 78.01$, which for $p = 1$ means $L_{3,1}^{10} = 1$, requiring a single binary variable $Z_{3,1,7,0}$, and $L_{3,1}^2 = 3$, requiring three binary variables ($Z_{3,1,1,0}, Z_{3,1,1,1}, Z_{3,1,1,2}$,) to approximate the number $7 \cdot 10^1 = (2^0 + 2^1 + 2^2) \cdot 10^1$.

Table 5. Results for (PR-MDT^{10}) followed by (P) as a function of discretization level p when discretizing over variables $Q_{i,t}$ and $V_{i,t}$ (without bound contraction)

	$Q_{i,t}$, $p = 0$				$Q_{i,t}$, $p = -1$				$V_{i,t}$, $p = 1$				$V_{i,t}$, $p = 0$				$V_{i,t}$, $p = -1$			
$|I|$	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs
2	209613	211053	0.63%	46.8	**209721**	210525	0.38%	3600[b]	209687	212170	1.15%	4.36	209687	210461	0.35%	138	209712	210161	0.21%	3600[b]
4	371584	375444	0.97%	3600[b]					371462	376800	1.32%	31.7	371765	376655	1.29%	3600[b]				
7	744150	759800	1.95%	3600[b]					744470	760905	2.10%	3600[b]								

[a] Optimality gap calculated using z from DICOPT. [b] Maximum resource limit.

Table 6. Results for (PR-MDT^{10}) and (P) preceded by bound contraction using (BC-MCR)

	$Q_{i,t}$, $p = 0$				$Q_{i,t}$, $p = -1$				$V_{i,t}$, $p = 1$				$V_{i,t}$, $p = 0$				$V_{i,t}$, $p = -1$			
$|I|$	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs
2	209613	210838	0.53%	22.2	**209721**	210371.4	0.31%	3600[b]	209687	211535	0.86%	3.22	209687	210290	0.27%	88.4	**209721**	210147	0.20%	3600[b]
4	371584	374794	0.80%	3600[b]					371331	375344	0.94%	22.4	371709	374556	0.73%	3600[b]				
7	744591	757260	1.62%	3600[b]					744256	757260	1.62%	3600[b]	744870	759425	1.90%	3600[b]				

[a] Optimality gap calculated using z from DICOPT. [b] Maximum resource limit.

Table 7. MILP problem statistics for (PR-MDT^{10}) as a function of discretization level p when discretizing over variables $Q_{i,t}$ and $V_{i,t}$, after bound contraction

	$Q_{i,t}$						$V_{i,t}$								
	Binary variables		Total variables		Equations		Binary variables			Total variables			Equations		
$|I|$	$p = 0$	$p = -1$	$p = 0$	$p = -1$	$p = 0$	$p = -1$	$p = 1$	$p = 0$	$p = -1$	$p = 1$	$p = 0$	$p = -1$	$p = 1$	$p = 0$	$p = -1$
2	244	724	860	2060	1755	3315	192	469	949	721	1436	2636	1581	2577	4137
4	444		1695		3621		382	927		1529	3149		3481	5895	
7	838		3187		6819		706			2869			6512		

Table 8. Results for (PR-MDT^2) and (P) preceded by bound contraction using (BC-MCR)

	$Q_{i,t}, p = 0$				$Q_{i,t}, p = -1$				$V_{i,t}, p = 1$				$V_{i,t}, p = 0$				$V_{i,t}, p = -1$			
$|I|$	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs	z [\$]	z^R [\$]	gap^a	CPUs
2	209613	210838	0.53%	30.6	**209721**	210493.6	0.37%	3600[b]	209687	211535	0.86%	3.21	209687	210290	0.27%	119	**209721**	210507	0.37%	3600[b]
4	371584	374783	0.79%	3600[b]					371331	375344	0.94%	23.2	**371812**	374381	0.69%	3600[b]				
7	744483	757901	1.71%	3600[b]					744410	757113	1.60%	3600[b]	744527	758533	1.79%	3600[b]				

[a] Optimality gap calculated using z from DICOPT. [b] Maximum resource limit.

Table 9. MILP problem statistics for (PR-MDT^2) after bound contraction

	$Q_{i,t}$						$V_{i,t}$									
	Binary variables		Total variables		Equations		Binary variables			Total variables			Equations			
$|I|$	$p = 0$	$p = -1$	$p = 0$	$p = -1$	$p = 0$	$p = -1$	$p = 1$	$p = 0$	$p = -1$	$p = 1$	$p = 0$	$p = -1$	$p = 1$	$p = 0$	$p = -1$	
2	270	594	921	1743	1925	3332	192	480	836	721	1441	2337	1581	2805	4333	
4	494		1823		3992		478	1104		1721	3409		3769	6737		
7	932		3445		7561		906			3315			7288			

5.5. Bound contraction using multiparametric disaggregation

A novel aspect of this paper has been to propose using multiparametric disaggregation in the bound contraction stage. The advantage is that by using a tighter MILP formulation than the one resulting from McCormick envelopes, we get an even smaller domain for the variables that make up the bilinear terms, which in turn will allow achieving smaller optimality gaps. The disadvantage comes from solving a larger MILP per iteration, due to the additional binary variables and constraints resulting from the discretization, leading to a larger computational time.

In Table 10, we show the results for the McCormick relaxation MILP (**PR-MC**) after performing bound contraction using (**BC-MDT**[10]) for three different settings in terms of discretization variables and accuracy level. When compared to the McCormick bound contraction in Table 4, it can be seen that we were able to reduce the gap from 2.02% to 0.86%, and from 2.08 to 1.86%, respectively for $|I| = 2,4$. However, an unreasonable amount of computational time was spent (considering that one is making decisions for the day ahead market) when using the highest accuracy level, at 17 and 173 h. So, do the tighter variable bounds really make a difference? The answer can be found in Table 11.

While there is only a small improvement for BARON, the most noticeable from 0.39% (see Table 3) to 0.29%, multiparametric disaggregation can reduce the gap from 0.20% to a mere 0.04% for $|I| = 2$. This is 1/6 of the gap that can be obtained by BARON without bound contraction up to 65000 CPUs (z^R=210234, z=209721) and so the answer to the question is yes, but just for small problems. This because the gaps returned by multiparametric disaggregation for $|I| = 4$ ($V_{i,t}, p = 0$) are very similar to those obtained by (**PR-MDT**[2]) with the standard bound contraction (see Table 8), which took 1/700 of the time. It is also important to note that BARON returns worse objective values than multiparametric disaggregation.

Table 10. Results for (PR-MC) after bound contraction with (BC-MDT^{10})

	$V_{i,t}, p = 1$			$Q_{i,t}, p = 0$			$V_{i,t}, p = 0$		
$|I|$	z^R [\$]	gap^a	CPUs	z^R [\$]	gap^a	CPUs	z^R [\$]	gap^a	CPUs
2	213145	1.61%	746	212571	1.34%	3374	211539	0.86%	61334[b]
4	379043	1.91%	34644[b]	378867	1.86%	624247[b]			

[a] Optimality gap calculated using z from DICOPT. [b] Maximum computational time per iteration=3600 CPUs.

Table 11. Results for (PR-MDT^{10}) and (P) after bound contraction with (BC-MDT^{10})

|I|	(BC-MDT^{10})	BARON			$Q_{i,t}, p = -1/0, |I| = 2/4$			$V_{i,t}, p = -1/0, |I| = 2/4$		
		z [$]	z^R [$]	gap^a	z [$]	z^R [$]	gap^a	z [$]	z^R [$]	gap^a
2	$V_{i,t}, p = 1$	209712	210584	0.41%	**209721**	210013	0.14%	**209721**	209986	0.13%
	$Q_{i,t}, p = 0$	209710	210515	0.38%	**209721**	209971	0.12%	**209721**	209959	0.11%
	$V_{i,t}, p = 0$	209719	210322	0.29%	**209721**	209795	0.04%	**209721**	209878	0.07%
4	$V_{i,t}, p = 1$	371185	375875	1.08%	371642	374527	0.73%	371765	374273	0.66%
	$Q_{i,t}, p = 0$	371183	375927	1.09%	371642	374391	0.69%	371758	374369	0.68%

^{a} Optimality gap calculated using z from DICOPT. CPUs=3600 in all runs.

5.6. Optimal operating profiles

Finally, we show in Figure 2 the optimal operating profiles for the 7-reservoir problem. Notice that the background pattern is related to electricity price, with dark green representing a low cost and dark rose a high cost within the 24-hour period. Not surprisingly, the reservoirs increase the storage capacity while the electricity is cheap, to get the most of it when the electricity cost is at its peak. Production then starts to decrease in the last 4 hours to get all the reservoirs to their initial state. Other relevant information is that there is a single startup for every plant.
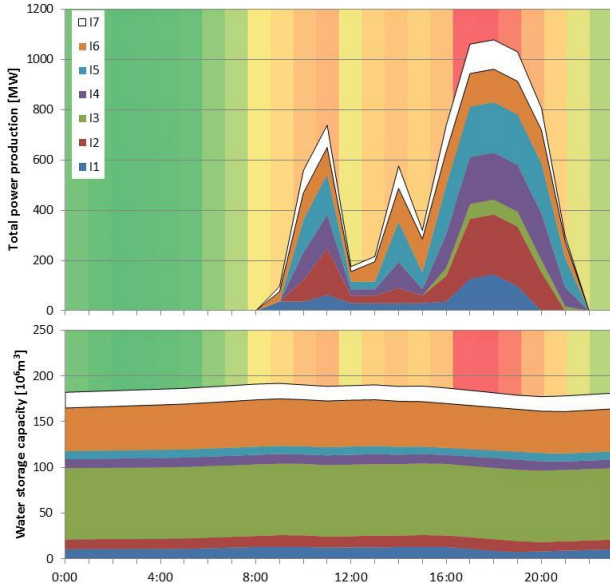
Figure 2: Optimal power and storage capacity profiles (profit= $744964)

6. Conclusions

This paper has shown that multiparametric disaggregation is capable of providing tighter bounds on the optimal solution for the hydro scheduling problem than those obtained by the state-of-the-art

global optimization algorithm BARON. It has also shown that the convex MINLP solver DICOPT is capable of providing better solutions to the problem in considerable less time.

A novel aspect has been the use of multiparametric disaggregation to further reduce the domain of the variables when compared to the standard approach using McCormick envelopes. The improved bounds come at a computational cost, which was shown to be worth for small problems. Acknowledging that multiparametric disaggregation offers a distinct treatment for the two variables defining the bilinear term, both discretization alternatives were tested, with the results showing no clear winner. When coupled with the accuracy level of the discretized variables, which should be decreased as the problem size increases, the overall recommendation is to test different settings to take the most of multiparametric disaggregation for the given computational resources.

Future work will look at the integration of multiparametric disaggregation with spatial branch-and-bound methods, taking advantage of the tighter relaxation to improve the bound on the objective function of the original problem, as well as to reduce the variables domain in bound contraction procedures. This will definitely lead to fewer iterations to reach a certain optimality gap, the question being if this strategy will be computationally more efficient than current ones.

Acknowledgments

References

[1] Catalão JPS, Pousinho HMI, Mendes VMF. Hydro energy systems management in Portugal: Profit-based evaluation of a mixed-integer nonlinear approach. Energy 2011; 36: 500-507.

[2] Catalão JPS, Mariano SJPS, Mendes VMF, Ferreira LAFM. Scheduling of head-sensitive cascaded hydrosystems: a nonlinear approach. IEEE Trans Power Syst. 2009; 24: 337-346.

[3] Catalão JPS, Mariano SJPS, Mendes VMF, Ferreira LAFM. Nonlinear optimization method for short-term hydro scheduling considering head-dependency. Euro Trans. Electr. Power 2010; 20: 172-183.

[4] Raman R, Grossmann IE. Relation between MILP modeling and logical inference for chemical process synthesis. Comp. Chem. Eng. 1991; 15: 7384.

[5] McCormick GP. Computability of global solutions to factorable nonconvex programs. Part I. Convex underestimating problems. Mathematical Programming 1976; 10: 147-175.

[6] Ruiz JP, Grossmann IE. Exploiting vector space properties to strengthen the relaxation of bilinear programs arising in the global optimization of process networks. Optimization Letters 2011; 5: 1-11.

[7] Bergamini ML, Aguirre P, Grossmann IE. Logic-based outer approximation for globally optimal synthesis of process networks. Computers and Chemical Engineering 2005; 29: 1914-1933.

[8] Karuppiah R, Grossmann IE. Global optimization for the synthesis of integrated water systems in chemical processes. Computers and Chemical Engineering 2006; 30: 650-673.

[9] Teles JP, Castro PM, Matos HA. Multiparametric disaggregation technique for global optimization of polynomial programming problems. Journal of Global Optimization 2013; 55: 227-251.

[10] Kolodziej S, Castro PM, Grossmann IE. Global Optimization of Bilinear Programs with a Multiparametric Disaggregation Technique. Journal of Global Optimization 2013; 57:1039-1063.

[11] Teles JP, Castro PM, Matos HA. Global Optimization of Water Networks Design using Multiparametric Disaggregation. Computers and Chemical Engineering 2012; 40: 132-147.

[12] Castro PM, Teles JP. Comparison of Global Optimization Algorithms for the Design of Water-using Networks. Computers and Chemical Engineering 2013; 52: 249-261.

[13] Balas E. Disjunctive Programming. Annals of Discrete Mathematics 1979; 5: 3-51.

[14] Balas E. Disjunctive Programming and a Hierarchy of Relaxations for Discrete Optimization Problems. SIAM Journal of Algebraic and Discrete Mathematics 1985; 6: 466-486.

[15] Kolodziej SP, Grossmann IE, Furman KC, Sawaya NW. A Discretization-based Approach for the Optimization of the Multiperiod Blend Scheduling Problem. Computers and Chemical Engineering 2013; 53: 122-142.

[16] García-González J, Parrilla E, Mateo A. Risk-averse profit-based optimal scheduling of a hydro-chain in the day-ahead electricity market. European Journal of Operational Research 2007; 181: 1354-1369.

[17] Duran MA, Grossmann IE. An outer-approximation algorithm for a class of mixed-integer nonlinear programs. Mathematical Programming 1986; 36 (3): 307-339.

[18] Ryoo HS, Sahinidis NV. A branch-and-reduce approach to global optimization. Journal of Global Optimization 1996; 8: 201-205.

[19] Carrión M, Arroyo JM. A Computationally Efficient Mixed-Integer Linear Formulation for the Thermal Unit Commitment Problem. IEEE Transactions on Power Systems 2006; 21 (3): 1371-1378.

[20] Quesada I, Grossmann IE. Global optimization of bilinear process networks with multicomponent flows. Computers & Chemical Engineering 1995; 19: 1219-1242.

[21] Meyer CA, Floudas CA. Global Optimization of a Combinatorially Complex Generalized Pooling Problem. AIChE Journal 2006; 52 (2): 1027-1037.

[22] Misener R, Thompson JP, Floudas CA. APOGEE: Global Optimization of Standard, Generalized, and Extended Pooling Problems via Linear and Logarithmic Partitioning Schemes. Computers & Chemical Engineering 2011; 35: 876-892.

[23] Jezowski J. Review of water network design methods with literature annotations. Industrial & Engineering Chemistry Research 2010; 49: 4475-4516.

[24] Faria DC, Bagajewicz MJ. A New Approach for Global Optimization of a Class of MINLP Problems with Applications to Water Management and Pooling Problems. AIChE Journal 2012; 58 (8): 2320-2335.

[25] Nápoles-Rivera F, Ponce-Ortega JM, El-Halwagi MM, Jiménez-Gutiérrez A. Global Optimization of Mass and Property Integration Networks with In-Plant Property Interceptors. Chemical Engineering Science 2010; 65: 4363-4377.

[26] Misener R, Floudas CA. GloMIQO: Global Mixed-Integer Quadratic Optimizer. Journal of Global Optimization 2013; 53, 3-50.

[27] Wicaksono DN, Karimi IA. Piecewise MILP Under- and Overestimators for Global Optimization of Bilinear Programs. AIChE Journal 2008; 54: 991-1008.

[28] Teles JP, Castro PM, Matos HA. Univariate Parameterization for Global Optimization of Mixed-Integer Polynomial Problems. European Journal of Operation Research 2013; 229: 613-625.

[29] Vielma JP, Ahmed S, Nemhauser G. Mixed-Integer Models for Nonseparable Piecewise-Linear Optimization: Unifying Framework and Extensions. Operations Research 2010; 58: 303-315.

[30] Vielma JP, Nemhauser G. Modeling Disjunctive Constraints with a Logarithmic Number of Binary Variables and Constraints. Math. Program. Ser. A 2011; 128: 49-72.

[31] Tawarmalani M, Sahinidis NV. Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming: Theory, Algorithms, Software and Applications. Boston, Ma: Kluwer Academic Publishers. 2002.

[32] Sherali H, Alameddine A. A New Reformulation-Linearization Technique for Bilinear Programming Problems. Journal of Global Optimization 1992; 2: 379-410.

[33] Liberti L, Pantelides CC. An Exact Reformulation Algorithm for Large Nonconvex NLPs involving bilinear terms. Journal of Global Optimization 2006; 36: 161-189.

[34] Rodriguez MA, Vecchietti A. A Comparative Assessment of Linearization Methods for Bilinear Models. Computers & Chemical Engineering 2013; 48: 218-233.

[35] Tsai J-F, Lin M-H. An Improved Framework for Solving NLIPs with Signomial Terms in the Objective or Constraints to Global Optimality. Computers & Chemical Engineering 2013; 53: 44-54.

[36] Harjunkoski I, Westerlund T, Pörn R, Skrifvars H. Different Transformations for Solving Non-convex Trim Loss Problems by MINLP. European Journal of Operational Research 1998; 105: 594-603.

[37] Zorn K, Sahinidis NV. Computational Experience with Applications of Bilinear Cutting Planes. Industrial & Engineering Chemistry Research 2013; 52: 7514-7525.

[38] Lima RM, Marcovecchio MG, Novais AQ, Grossmann IE. On the Computational Studies of Deterministic Global Optimization of Head Dependent Short-Term Hydro Scheduling. IEEE Transactions on Power Systems 2013. 28 (4):4336-4347.