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SYMBOLIC COMPUTATION AND THE RAYLEIGH-BÉNARD STABILITY PROBLEM

Giestas[1], M. C., Pina[2], H. L.

(1) LNEG
Estrada do Paço do Lumiar, 22, 1649-038 Lisboa
margarida.giestas@lneg.pt
(2) IDMEC-IST
Av. Rovisco Pais, 1049-001 Lisboa

Abstract

This paper analyzes the linear stability of an horizontal layer of fluid consisting of a mixture of water and salt. The layer is hotter at the bottom and cooler at the top thus having a tendency to destabilize. To counteract this a salt concentration gradient (denser at the bottom and lighter at the top) is sometimes present, either naturally as in the ocean or created artificially as in solar ponds.

The relevant governing equations are the linearized continuum mechanics balance laws applied to an incompressible, heat-conducting and salt-diffusing fluid, leading to a system of partial differential equations, from which the stability of a given base state has to be assessed with respect to arbitrary initial perturbations.

This problem involves intensive symbolic computations that can be much facilitated by the use of a Computer Algebra System (CAS).

Keywords: Linear stability, Rayleigh-Bénard problem, solar ponds, symbolic computation

1 Introduction

The present paper considers a horizontal layer of fluid consisting of a mixture of water and salt heated at the top by solar radiation. This situation is present naturally in the ocean or is artificially created as in solar ponds.

The temperature gradient (hotter at the bottom and cooler at the top) thus established tends to destabilize the fluid and a salt concentration gradient (denser at the bottom and lighter at the top) is meant to counteract its effects.

A solar pond is a basin of water intended to store solar energy. A temperature profile (higher temperature at the bottom and lower temperature at the top) is established due to solar radiation absorption and the salt concentration gradient (denser at the bottom and lighter at the top) acting in the opposite way is artificially created to prevent convective motions that would otherwise tend to homogenize the temperature and thus promote the return of the stored energy to the outside ambient destroying the solar pond very purpose. Thus the knowledge of the stability/instability conditions is crucial for the correct design and operation of these devices.

It is therefore of theoretical and practical interest to determine the circumstances for the onset of instability to occur, a problem known in the literature as a two-component (water and salt) Rayleigh-Bénard stability problem.

For previous work connected to the present topic see (Bénard, 1900), (Rayleigh, 1916), (Pellew and Southwell, 1940), (Reid, 1965), (Chandrasekhar, 1961a) (Stern, 1960), (Chandrasekhar, 1961b) and (Drazin and Reid, 2004) for a classical analysis. Extensive reviews can be found in (Geitling, 1998), (Koschmieder, 1993) and (Lappa, 2010). Other more practically oriented studies are (Dake and Harleman, 1969), (Akbarzadeh and Ahmadi, 1980), (Schechter et al., 1981a), (Sani, 1964), (Schechter et al., 1981b), (Platten and Legros, 1983), (Turner, 1985), (Costa et al., 1981), (Mukutmoni and Yang, 1993a), (Mukutmoni and Yang, 1993b), (Proctor, 1981), (Nield et al., 1993), (Cha et al., 1982), (Meyer et al., 1982), (Veronis, 1965), (Giestas et al., 1996) and (Giestas et al., 1997).

The mathematical model adopted in this paper assumes the fluid to be newtonian incompressible, conducts heat according to Fourier's law and the salt diffusion obeys Fick's law. An uniform gravitational field acts on the fluid body.

To account for the effect of temperature and salt concentration on the fluid density, the Boussinesq hypothesis is adopted: the specific mass appearing in the left hand sides of the relevant equations is supposed constant and equal to some average value whereas in the right hand sides it is assumed to vary linearly with temperature and salt concentration.

The stability study is performed by considering a certain equilibrium or base state and imposing sufficiently small perturbations upon it in such way that nonlinear terms present in the equations can be neglected leading to what is commonly called a linear stability analysis.

Due to the complexity of the governing equations the algebra involved in their solution is overwhelming and the purpose of the present paper is to show how a Computer Algebra System can be helpful in this regard.

The plan of the paper is as follows: in section 2 the relevant governing equations are presented, section 3 develops a perturbation analysis with respect to some base state (the steady state), in section 4 the various stability regimes of this state are determined in terms of some nondimensional parameters, section 5 elaborates on the use of a CAS in the context of this particular problem and finally section 6 collects some comments and conclusions.

Notation

A cartesian coordinate system is employed throughout with position given by $\mathbf{x} = (x, y, z)$ and time is denoted by t. The symbols ∇, $\nabla\cdot$ and $\Delta \equiv \nabla\cdot\nabla$ stand for the gradient, the divergence and the laplacian respectively.

The layer of fluid has depth H in the vertical direction z and is supposed to extend to infinity in both horizontal directions x and y. For simplicity it is assumed that the problem is bidimensional, the coordinate y playing no role. Moreover the variations along x are supposed periodic with wavelength $\Lambda > 0$. Thus the problem domain amounts to an open rectangle $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ (see Figure 1).

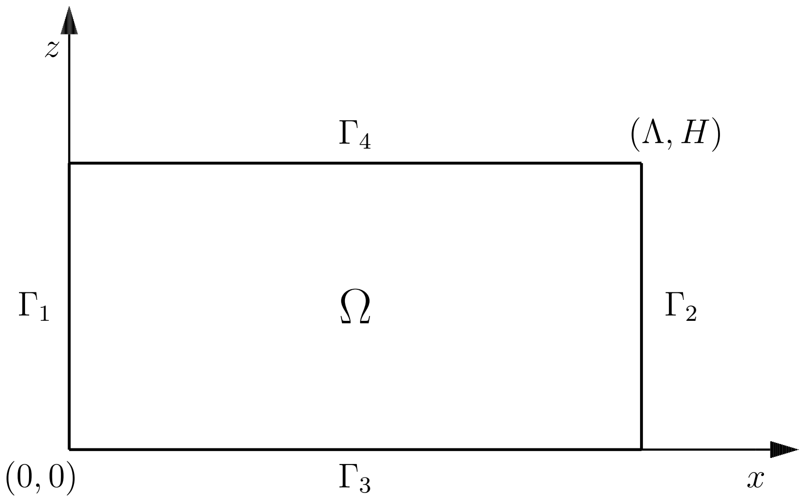

Figure 1: Geometry and notation

Additionally, we employ the following notations: θ is the temperature, σ the salt concentration and $\mathbf{u}$ the velocity, p the pressure, $\mathbf{b}$ is the specific buoyancy force. The following material properties are assumed: K_θ is the heat diffusivity, K_σ the salt diffusivity, ν the kinematic viscosity, all supposed constant and $\rho = \rho(\theta, \sigma)$ is the specific mass. The gravity acceleration is denoted by $\mathbf{g} = (0, -g)$.

2 The governing equations

The governing equations are the standard conservation laws for continuum media. The water-salt solution is assumed to behave as a newtonian fluid conducting heat according to Fourier's law and the salt diffusion is governed by Fick's law.

Also the Boussinesq hypothesis is adopted: the specific mass appearing in the left hand sides of the relevant equations is supposed constant and equal to some average value ρ_0 whereas in the right hand sides it is assumed to vary with temperature and salt according to some relation $\rho = \rho(\theta, \sigma)$.

We present the relevant equations whose derivations are well known and can be looked for at any fluid dynamics textbook, see (Gurtin et al., 2010) for instance.

The balance of energy yields the equation:

$$\partial_t\theta + (\mathbf{u}\cdot\nabla)\theta = K_\theta\,\Delta\,\theta \tag{1}$$

The salt diffusion yields the equation:

$$\partial_t\sigma + (\mathbf{u}\cdot\nabla)\sigma = K_\sigma\,\Delta\,\theta \tag{2}$$

The linear momentum equation delivers the Navier-Stokes system:

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\,\Delta\,\mathbf{u} - \nabla p + \mathbf{b} \tag{3}$$

with the buoyancy force given by the Boussinesq approximation

$$\mathbf{b} = (1 - \alpha(\theta - \theta_0) + \beta(\sigma - \sigma_0))\mathbf{g}$$

where $\alpha \geq 0$ and $\beta \geq 0$ are expansion coefficients for temperature and salinity and θ_0 an σ_0 are some reference values.

The balance of mass is expressed by the usual incompressibility condition

$$\nabla \cdot \mathbf{u} = 0 \tag{4}$$

As can be seen, our mathematical model consists of a system of coupled partial differential equations, (1) and (2), comprising two time dependent advection-diffusion equations and the time dependent Navier-Stokes equation (3) plus the incompressibility condition (4). This amounts to a Rayleigh-Bénard problem with two components (temperature and salt concentration).

3 Perturbation analysis

Given an equilibrium state that we denote by $(\bar{\theta}, \bar{\sigma}, \bar{\mathbf{u}}, \bar{p})$, the stability problem consists in determining if some arbitrary perturbations $(\theta', \sigma', \mathbf{u}', p')$ will grow or decay, leading respectively to instability or stability of the given equilibrium state. We therefore write each variable as the sum of the respective equilibrium state value plus its perturbation, that is,

$$\begin{aligned} \theta &= \bar{\theta} + \theta' \\ \sigma &= \bar{\sigma} + \sigma' \\ \mathbf{u} &= \bar{\mathbf{u}} + \mathbf{u}' \\ p &= \bar{p} + p' \\ \mathbf{b} &= \bar{\mathbf{b}} + \mathbf{b}' \end{aligned} \tag{5}$$

The equilibrium state we consider is the steady state defined by

$$\partial_t \bar{\theta} = 0, \quad \partial_t \bar{\sigma} = 0, \quad \partial_t \bar{\mathbf{u}} = 0, \quad \bar{\mathbf{u}} = 0 \tag{6}$$

and imposing all the variations to be confined to the z direction.

Therefore, from (1), (2) and (3), we have that

$$\begin{aligned} K_\theta \, \Delta \, \bar{\theta} &= 0 \\ K_\sigma \, \Delta \, \bar{\sigma} &= 0 \\ -\nabla \bar{p} + \bar{\mathbf{b}} &= 0 \end{aligned} \tag{7}$$

with the buoyancy force $\bar{\mathbf{b}}$ and its perturbation $\mathbf{b}'$ given by

$$\bar{\mathbf{b}} = (1 - \alpha(\bar{\theta} - \theta_0) + \beta(\bar{\sigma} - \sigma_0))\mathbf{g} \tag{8a}$$

$$\mathbf{b}' = (-\alpha\theta' + \beta\sigma')\mathbf{g} \tag{8b}$$

The solutions of equations (7) are easy to obtain, resulting in

$$\begin{aligned} \bar{\theta} &= \theta_3 - \Delta\theta \frac{z}{H}, \qquad \Delta\theta = \theta_3 - \theta_4 \\ \bar{\sigma} &= \sigma_3 - \Delta\sigma \frac{z}{H}, \qquad \Delta\sigma = \sigma_3 - \sigma_4 \end{aligned} \tag{9}$$

The equation for $\bar{p}$ also offers no difficulty but having no particular interest in the sequel we omit its solution.

Introducing relations (5) in equations (1), (2) and (3) we get the following partial differential equations for the perturbations

$$\begin{aligned} \partial_t \theta' + (\mathbf{u}' \cdot \nabla)\bar{\theta} + (\mathbf{u}' \cdot \nabla)\theta' &= K_\theta \, \Delta \, \theta' \\ \partial_t \sigma' + (\mathbf{u}' \cdot \nabla)\bar{\sigma} + (\mathbf{u}' \cdot \nabla)\sigma' &= K_\sigma \, \Delta \, \sigma' \\ \partial_t \mathbf{u}' + (\mathbf{u}' \cdot \nabla)\mathbf{u}' &= -\nabla p' + \nu \, \Delta \, \mathbf{u}' + \mathbf{b}' \\ \nabla \cdot \mathbf{u}' &= 0 \end{aligned} \tag{10}$$

As we are concerned with *linear* stability we drop the nonlinear terms in (10), thus getting the following system of linear partial differential equations

$$\partial_t \theta' + (\mathbf{u}' \cdot \nabla)\bar{\theta} = K_\theta \, \Delta \, \theta' \tag{11a}$$

$$\partial_t \sigma' + (\mathbf{u}' \cdot \nabla)\bar{\sigma} = K_\sigma \, \Delta \, \sigma' \tag{11b}$$

$$\partial_t \mathbf{u}' = -\nabla p' + \nu \, \Delta \, \mathbf{u}' + \mathbf{b}' \tag{11c}$$

$$\nabla \cdot \mathbf{u}' = 0 \tag{11d}$$

By expressing the velocity $\mathbf{u}'$ as

$$\mathbf{u}' = \nabla \times \boldsymbol{\psi}' \tag{12}$$

the incompressibility equation (11d) becomes automatically satisfied. The potential vector $\boldsymbol{\psi}'$, for the present two dimensional case, assumes the form

$$\boldsymbol{\psi}' = (0, -\psi'(x,z,t), 0) \tag{13}$$

for some stream function ψ'. The velocity is thus given by

$$\mathbf{u}'_x = \partial_z \psi', \qquad \mathbf{u}'_y = 0, \qquad \mathbf{u}'_z = -\partial_x \psi' \tag{14}$$

The pressure p' can be eliminated from equation (11c) by applying the $\nabla \times$ operator to both its members, yielding after using relation (8b) and some simplifications, the expression

$$\partial_t \Delta \psi' = \nu \Delta^2 \psi' - \alpha \nabla\theta' \times \mathbf{g} + \beta \nabla \sigma' \times \mathbf{g} \tag{15}$$

This is a fourth order (in space) partial differential equation that has to be complemented by some appropriate boundary conditions to be specified below.

In order to proceed it is useful to nondimensionalize equations (11a), (11b) and (15). Omitting the details (see (Giestas et al., 1996)) and keeping the same symbols for the nondimensional variables to avoid notational proliferation, we get

$$\partial_t \theta' + (\mathbf{u}' \cdot \nabla)\bar{\theta} = \Delta \theta' \tag{16a}$$

$$\partial_t \sigma' + (\mathbf{u}' \cdot \nabla)\bar{\sigma} = \tau \Delta \sigma' \tag{16b}$$

$$\partial_t \Delta \psi' = Pr \Delta^2 \psi' - Pr R_\theta \nabla\theta' \times \mathbf{g} - Pr R_\sigma \nabla \sigma' \times \mathbf{e_z} \tag{16c}$$

where

$$R_\theta = \frac{\alpha g H^3 \Delta\theta}{\nu K_\theta}, \quad R_\sigma = \frac{\beta g H^3 \Delta\sigma}{\nu K_\theta}, \quad Pr = \frac{\nu}{K_\theta}, \quad \tau = \frac{K_\sigma}{K_\theta} \tag{17}$$

are nondimensional numbers, respectively, the temperature Rayleigh number, the salinity Rayleigh number, the Prandtl number and the diffusivities ratio and $\mathbf{e}_z = (0,0,1)$ is the unit basis vector of the z axis.

Boundary conditions

We need to supply the system of partial differential equations (16a), (16b) and (16c) with appropriate boundary conditions. Take into account the steady state fields, the following conditions for θ' and σ' are adopted:

$$\partial_x \theta' = 0 \quad \text{on} \quad \Gamma_1 \tag{18a}$$

$$\partial_x \sigma' = 0 \quad \text{on} \quad \Gamma_2 \tag{18b}$$

$$\theta' = 0 \quad \text{on} \quad \Gamma_3 \tag{18c}$$

$$\sigma' = 0 \quad \text{on} \quad \Gamma_4 \tag{18d}$$

The physical meaning of these relations should be clear: (18a) and (18b) impose the fluxes to be zero on the lateral boundaries Γ_1, Γ_2 while (18c) and (18d) impose the perturbations to vanish on the bottom and top boundaries Γ_3, Γ_4.

Additionally, we require that no fluid leaves the domain Ω, that is, $\partial\Omega$ is a streamline, and the velocity on boundaries Γ_3 and Γ_4 be zero (the so called rigid-rigid condition). Taking relations (14) into account this in turn implies the following additional conditions

$$\psi' = 0 \quad \text{on} \quad \partial\Omega \tag{19a}$$

$$\partial_z \psi' = 0 \quad \text{on} \quad \Gamma_3 \tag{19b}$$

$$\partial_z \psi' = 0 \quad \text{on} \quad \Gamma_4 \tag{19c}$$

4 Stability analysis

Now we want to investigate which perturbation modes lead to instability. Considering the boundary conditions and the periodicity in the x direction we stipulate the following mode

$$\theta' = a_\theta \exp(st)\phi_\theta(x,z), \qquad \phi_\theta(x,z) = \cos(2\pi x/\Lambda)\sin(\pi z/H) \tag{20a}$$

$$\sigma' = a_\sigma \exp(st)\phi_\sigma(x,z), \qquad \phi_\sigma(x,z) = \cos(2\pi x/\Lambda)\sin(\pi z/H) \tag{20b}$$

$$\psi' = a_\psi \exp(st)\phi_\psi(x,z), \qquad \phi_\psi(x,z) = \sin(2\pi x/\Lambda)\sin^2(\pi z/H) \tag{20c}$$

where $a_\theta, a_\sigma, a_\psi \in \mathbb{R}$ are the amplitudes and $s \in \mathbb{C}$ is the growth rate parameter determining the stability of this mode. Figure 2 depicts the kind of flow perturbation we are concerned about.

The functions above satisfy the prescribed boundary conditions but not the partial differential equations (16a), (16b) and (16c). Therefore we attempt to obtain an approximate solution by the Galerkin method: we take the residual of each equation

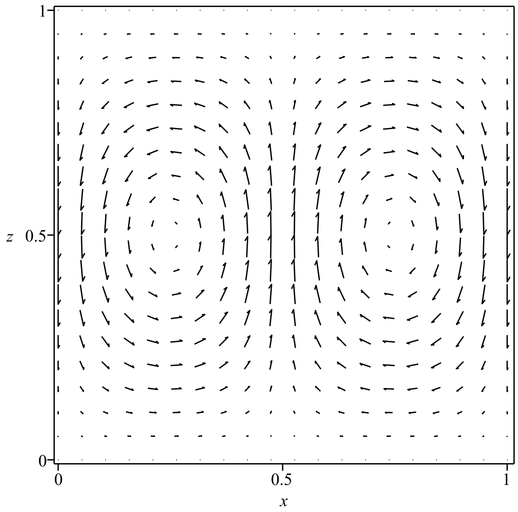

Figure 2: Perturbation flow (depicted for the unit square)

and orthogonalize it with respect to the respective base function. Exemplifying with the equation for temperature (16a), we have the residual

$$r_\theta = a_\theta \exp(st)(s\phi_\theta - \Delta\,\phi_\theta) + a_\psi \exp(st)(\nabla \times \boldsymbol{\phi_\psi}) \cdot \bar{\theta}$$

which we then orthogonalize

$$(r_\theta, \phi_\theta) \equiv \int_0^\lambda \int_0^1 r_\theta \phi_\theta \, \mathrm{d}x \, \mathrm{d}z = 0 \tag{21}$$

where now $\lambda = \Lambda/H$.

We proceed similarly for the salinity σ' and stream function ψ'.

After carrying out the required algebra, the following algebraic system is obtained for the amplitudes

$$(\mathbf{A} - s\mathbf{M})\mathbf{a} = \mathbf{0}, \qquad \mathbf{a} = \begin{pmatrix} a_\theta \\ a_\sigma \\ a_\psi \end{pmatrix} \tag{22}$$

where matrix $\mathbf{M}$ (the mass matrix) is diagonal and invertible.

In order for this system to have nontrivial solutions, i.e., $\mathbf{a} \neq 0$, s has to be an eigenvalue of the generalized eigenvalue problem (22). Equivalently, s must be an eigenvalue of the matrix

$$\mathbf{S} = \mathbf{M}^{-1}\mathbf{A} = \begin{pmatrix} -c & 0 & -e \\ 0 & -\tau c & -e \\ -fP_\theta & fP_\sigma & -d \end{pmatrix} \tag{23}$$

where

$$\begin{aligned} P_\theta = PrR_\theta \geq 0, \qquad P_\sigma &= PrR_\sigma \geq 0 \\ c = c(\lambda) &\equiv \frac{\pi^2\left(\lambda^2+4\right)}{\lambda^2} > 0 \\ d = d(\lambda) &\equiv \frac{4\pi^2\left(\lambda^4+2\,\lambda^2+3\right)}{\lambda^2\left(\lambda^2+3\right)} > 0 \\ e = e(\lambda) &\equiv \frac{16}{3\lambda} > 0 \\ f = f(\lambda) &\equiv \frac{16\lambda}{3\pi^2\left(\lambda^2+3\right)} > 0 \end{aligned} \tag{24}$$

The eigenvalues s_i, $i = 1, 2, 3$ of $\mathbf{S}$ are the zeros of the characteristic polynomial

$$p(s) = \det(s\mathbf{I} - \mathbf{S}) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{25}$$

its coefficients being given by

$$\begin{aligned} a_0 &= -ecf\tau P_\theta + ecfP_\sigma + c^2 d\tau \\ a_1 &= -efP_\theta + efP_\sigma + c(c+d)\tau + dc \\ a_2 &= d + c(\tau+1) > 0 \end{aligned} \tag{26}$$

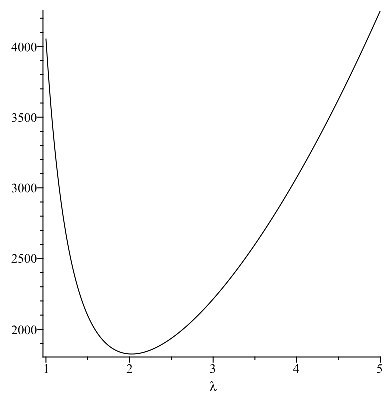

Figure 3: The group $\frac{cd}{ef}$ in the right hand side of (30) as a function of λ

Asymptotic stability

In order to have asymptotic stability the eigenvalues s_i of $\mathbf{S}$ must lie in the complex left half-plane, that is, $\mathrm{Re}(s_i) < 0$ for $i = 1, 2, 3$, or, in other words, for the characteristic polynomial $p(s)$ to be stable. For this to happen, the Routh-Hurwitz criterion (see (Gradshteyn and Ryzhik, 1980) or (Bernstein, 2005)) must be obeyed

$$a_0 > 0, \quad a_1 > 0, \quad a_2 > 0 \tag{27a}$$
$$a_1 a_2 - a_0 > 0 \tag{27b}$$

Since in this problem the last relation in (27a) is automatically satisfied we are left with the three conditions

$$a_0 > 0, \quad a_1 > 0 \tag{28a}$$
$$a_1 a_2 - a_0 > 0 \tag{28b}$$

which lead to the following simultaneous inequalities

$$P_\theta < \frac{1}{\tau} P_\sigma + \frac{cd}{ef} \tag{29a}$$
$$P_\theta < P_\sigma + \frac{c(c+d)\tau}{ef} + \frac{cd}{ef} \tag{29b}$$
$$P_\theta < \frac{c\tau + d}{c+d} P_\sigma + \frac{c^2\tau^2}{ef} + \frac{c(c+d)\tau}{ef} + \frac{cd}{ef} \tag{29c}$$

Given that $0 < \tau < 1$ and all the terms in the right hand sides are positive, these three conditions combine to give

$$P_\theta < \frac{c\tau + d}{c+d} P_\sigma + \frac{cd}{ef}, \quad \forall \lambda \tag{30}$$

The group $\frac{c\tau+d}{c+d}$ increases with λ thus having a minimum at $\lambda = 0$ which evaluates to $(1+\tau)/2$. The group $\frac{cd}{ef}$ depends on λ in such a way that it possesses a minimum which can be found to occur at

$$\lambda \approx \lambda_c = 2.03 \tag{31}$$

and evaluates to 1824.7 as shown in Figure 3. In this case, linear asymptotic stability is guaranteed if

$$P_\theta < 0.5(1+\tau)P_\sigma + 1824.7 \tag{32}$$

The values 2.03 in (31) and 1824.7 in (32) compare well with those reported in (Chandrasekhar, 1961b) (2.016 and 1708 respectively for $P_\sigma = 0$).

Oscillatory regime

A question that has some interest is the possibility of a purely oscillatory regime, that is, the existence of two purely imaginary eigenvalues $s_2 = \mathrm{i}\,\omega$, $s_3 = -\mathrm{i}\,\omega$ with $\omega > 0$ together with a negative eigenvalue $s_1 < 0$. In that case, introducing these facts in the characteristic polynomial (25) yields

$$p(s) = (s - s1)(s - \mathrm{i}\,\omega)(s + \mathrm{i}\,\omega) = 0 \tag{33}$$
$$= s^3 - s_1 s^2 + \omega^2 s - s_1 \omega^2 \tag{34}$$

Then, we must have

$$s_1 = -a_2 < 0 \tag{35a}$$
$$\omega^2 = a_1 > 0 \tag{35b}$$
$$a_0 = -s_1\omega^2 \tag{35c}$$

Relation (35a) is automatically verified and (35c) is equivalent to $a_0 = a_1 a_2$, a constraint on the characteristic polynomial coefficients that, for given values of P_θ and P_σ, yields the values of $\lambda > 0$ (if any) for which an oscillatory regime can exist. Therefore, we are left with

$$a_1 > 0 \tag{36a}$$
$$a_0 = a_1 a_2 > 0 \tag{36b}$$

From these two relations we obtain

$$P_\theta = \frac{\tau c + d}{c+d} P_\sigma + \frac{c^2}{ef}\tau^2 + \frac{c(c+d)}{ef}\tau + \frac{cd}{ef} \tag{37a}$$
$$P_\sigma > \frac{c(c+d)}{ef}\frac{\tau^2}{1-\tau} \tag{37b}$$

The right hand side of relation (37b) possesses a minimum that occurs at $\lambda = \lambda_c \approx 2.23$. For this value, the above relations become

$$P_\sigma > 2725.5\frac{\tau^2}{1-\tau} \tag{38a}$$
$$P_\theta = (0.322 + 0.678\tau)P_\sigma + 878.2\tau^2 + 2725.5\tau + 1847.3 \tag{38b}$$

which determine the region where a oscillatory regime takes place.

Steady-state regime

Another particular case is the steady state regime defined by $\mathrm{Re}(s_1) < 0$, $\mathrm{Re}(s_2) < 0$ and one zero eigenvalue, $s_3 = 0$. From (25) one obtains

$$a_0 = 0, \quad a_1 > 0, \quad a_2 > 0 \tag{39}$$

Since $a_2 > 0$ automatically, these relations imply that

$$P_\theta = \frac{1}{\tau}P_\sigma + \frac{cd}{ef} \tag{40a}$$
$$P_\sigma < \frac{\tau^2}{1-\tau}\frac{c(c+d)}{ef} \tag{40b}$$

The right hand side of relation (40b) possesses a minimum that occurs at $\lambda = \lambda_c \approx 2.23$. For this value, the above relations become

$$P_\theta = \frac{1}{\tau}P_\sigma + 1847.3 \tag{41a}$$
$$P_\sigma < 2725.5\frac{\tau^2}{1-\tau} \tag{41b}$$

These relations determine the region where a steady-state regime takes place.

5 The role of CAS

We expand here on the advantages on the use of a CAS to perform some of the computations outlined above.

Firstly the basis functions need to be setup, the required derivatives computed and the boundary conditions they have to obey must be checked in order to proceed.

The determination of the stability matrix $\mathbf{S}$ coefficients introduced in expression (23) requires the systematic evaluation of integrals typified by equation (21). This amounts to compute many two-dimensional integrals over a rectangle

$$\int_0^\lambda \int_0^1 f(x,z)\,\mathrm{d}x\,\mathrm{d}z \tag{42}$$

where f depends on the basis functions and their derivatives leading eventually to rather complicated expressions. Furthermore the resulting integral values need to be properly simplified in order to enhance their readability and provide insight.

The next task is to compute the stability matrix $\mathbf{S}$ characteristic polynomial (25) whose roots are the eigenvalues that decide the type of stability/instability to be expected. This envolves the Routh-Hurwitz criteria and a discussion whether these eigenvalues have negative real parts leading to asymptotic stability or not.

The stability conditions thus obtained depend on some parameters, namely the wavelength λ, see for instance relation (37b), and we wish to know what its most critical values are. This amounts to find the minimum of certain functions (Figure 3 shows a typical example) a task that albeit not intrinsically difficult results in extensive algebraic manipulations.

An important benefit accruing in using a CAS is that once the program is developed there is room for further experimentation with minimal investment, for instance trying other basis functions and boundary conditions to assess their soundness and influence on the final results.

6 Conclusions

The problem described is challenging from the algebraic point of view thus presenting many opportunities for the use of a CAS system. We list here the tasks where we actually employed such a CAS (Maple):

- derivation of the steady state solutions (9) that is easy in this case but could become more complicated if additional factors, e.g. heat sources, were taken into account;
- computation of the matrices $\mathbf{M}$, $\mathbf{A}$, expression (22), and the stability matrix $\mathbf{S}$ introduced in expression (23), requiring the evaluation of many symbolic integrals;
- obtention of the characteristic polynomial (25) and the respective roots (eigenvalues);
- several easy but very tedious minimization problems with respect to the wavelength λ.

In addition, once the CAS program is in place, variants to the problem treated, for instance, change of basis functions or boundary conditions, can be easily tried and evaluated.

In any case, having recourse to a CAS should not be understood as a brute force method and some ingenuity from the part of the user seems to be indispensable to obtain meaningful results.

Acknowledgments

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